

A SIMPLIFIED ANALYSIS METHOD FOR WALL OUT OF PLANE ANCHORAGE FORCES

EXECUTIVE SUMMARY

This report presents the results of a preliminary study on wall anchorage forces in certain type of PG&E sub-station buildings. This study is based upon PG&E Substation K, located in San Francisco, CA. Substation K represents a typical substation building built around the 1920's. The substation structures are typically reinforced concrete shear wall structures with a complete gravity structural steel frame.. They have a partial second floor which is referred to as gallery level. Their lateral load resisting system comprises roof and gallery levels acting as diaphragms and shear walls as the main system to transfer the lateral loads to the foundation.

Two separate analyses have been performed. A three dimensional dynamic model using SAP200 and a simplified multi-degree of freedom spring model. Four different variables and their effect on wall out of plane anchorage forces are studied in this report. These variables are:

1. Wall to diaphragm connection (fixed and pinned)
2. Wall out of plane stiffness
3. Wall foundation rocking
4. Diaphragm stiffness
5. Diaphragm relative stiffness

It is found that the wall out of plane stiffness, diaphragm stiffness, and relative stiffness of two diaphragms have significant effect on the wall out of plane anchorage forces. By including the wall out of plane stiffness, the anchorage forces reduced by 20% to 30% as compared to the FEMA 356 forces. The gallery floor diaphragm, due to large opening in the middle, behaves as flexible diaphragm, whereas, the roof diaphragm behaves as a rigid diaphragm. The analyses show that the wall out of plane anchorage forces based upon the diaphragm, are on the order of 25% less than the corresponding FEMA 356 forces. It is also found that the wall foundation rocking and wall to diaphragm connection (fixed or pinned) does not have significant impact on the wall out of plane anchorage forces.

Recommendation are presented on a proposed design equation with the above noted parameters. Further studies are recommended to study the proposed equation and the parameters.

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APPENDIX A

1.0 INTRODUCTION

The EQE Structural Engineers Division of ABSG Consulting Inc. (ABS Consulting) has performed a preliminary study on wall anchorage forces in certain type of PG&E substation buildings. This study is based upon PG&E Substation K, located in San Francisco, CA. Substation K represents a typical substation building built around the 1920's. The substation structures are typically reinforced concrete shear wall structures with a complete gravity structural steel frame.. They have a partial second floor which is referred to as gallery level. Their lateral load resisting system is comprised of roof and gallery levels acting as diaphragms and shear walls as the main system to transfer the lateral loads to the foundation.

For the substation K building ABS Consulting has performed a parametric study on the wall anchorage forces based on a detailed mathematical model using program SAP2000 as well as on a simplified six degree of freedom spring model. The intent of this study is to propose simplified design equations that can be used in a typical design office for the analysis and design of the wall out of plane anchorage. The proposed equations will be similar to the FEMA 356 and UBC97 equations, but will include additional variables such as out of plane wall stiffness, diaphragm stiffness and foundation rocking effects.

A simplified multi-degree of freedom analysis method was proposed by Roeder, et. al., in their report published by Roeder, et. al., titled Evaluation of the Seismic Vulnerability of Substation Buildings. This study is based upon the Roeder model. However, additional parameters as noted in the executive summary have been studied in this report and a design equation is proposed.

2.0 ANALYSIS METHODOLOGY

Two separate models are used for analysis. A three dimensional dynamic model using SAP2000 and a simplified multi-degree of freedom (MDOF) spring model.

2.1 3-D Linear Dynamic Analysis (SAP2000)

The walls as well as the roof and the gallery level of the building have been modeled using shell elements with appropriate mass and stiffness characteristics. The interior gravity steel frame has been modeled using 3D linear frame elements. The out-of-plane wall was connected to the diaphragms with rigid truss elements representing the wall anchors.

The following analyses have been performed using the SAP2000 model:

- Response spectrum analysis have been employed to compare the resulting out-of-plane wall anchorage forces with the recommended design level forces both at the roof and gallery levels of the building; the effects of out-of-plane wall stiffness and end connections have been investigated;
- Time history analysis have been employed to compare the resulting out-of-plane wall anchorage forces with the recommended design level forces at gallery level of the building; the effect of out-of-plane wall rocking has been investigated;

2.2 MDOF Simplified Spring Model

The building has been modeled as two story structure, with six degrees of freedom as shown in Figure 2-1:

- DOF 1 – degree of freedom at the in plane wall at the gallery level
- DOF 2 – degree of freedom at the in plane wall at the roof level
- DOF 3 – degree of freedom at the midpoint of the gallery diaphragm
- DOF 4 – degree of freedom at the midpoint of the roof diaphragm
- DOF 5 – degree of freedom at the out-of-plane wall at the gallery level
- DOF 6 – degree of freedom at the out-of-plane wall at the roof level

The mass and stiffness of the in-plane shear walls have been incorporated into the model as one dimensional elements; consistent mass and stiffness matrices have been derived for roof and gallery level diaphragms, the effect of the large opening at the center of the roof diaphragm has been accounted for in the analysis; out-of-plane wall has been modeled as a seven node shell element as shown in Figure 2-2. Sample shape functions used in deriving consistent mass and stiffness properties of the out-of-plane wall are shown in Figure 2-3. The total of seven shape function for the out-of-plane wall have been obtained by modeling the deformed shape of the wall by assigning unit displacement at one of the nodes shown in Figure 2-3a, while restraining all other nodes. Using those shape functions (i,j) -th element mass and stiffness matrices of the out-of-plane wall have been computed using the following formulations based on theory of plate vibration:

$$m_{ij} = \bar{m} \int_d^L \int_d^H \psi_i(x, z) \psi_j(x, z) dx dz$$

$$k_{ij} = \int_0^L \int_0^H \left[M_{xi}(x, z) \left(-\frac{d^L}{dz^2} \psi_j(x, z) \right) + M_{zi}(x, z) \left(-\frac{d^L}{dx^2} \psi_j(x, z) \right) + LM_{xi}(x, z) \left(\frac{d^L}{dx dz} \psi_j(x, z) \right) \right] dz dx$$

where

$$M_{xi}(x, z) = -D \left(v \frac{d}{dx^2} \psi_i(x, z) + \frac{d}{dz^2} \psi_i(x, z) \right)$$

$$M_{zi}(x, z) = -D \left(\frac{d^L}{dx^2} \psi_i(x, z) + v \frac{d^L}{dz^2} \psi_i(x, z) \right)$$

$$M_{xzi} = D(1-v) \frac{d^L}{dx dz} \psi_i$$

In the above equations:

m_{ij} and k_{ij} - is the (i, j) -th elements of the mass and stiffness matrices, respectively;

\bar{m} - is the mass of the wall per unit area;

ψ_i and ψ_j - are the i -th and j -th shape functions;

L and H - are the length and height of the wall, respectively;

M_{xi} , M_{zi} and M_{xzi} - are the bending moments around the respective axis, when the wall is deflected according to the i -th shape function;

$D = \frac{E t_w^3}{12(1-v)}$ - is the flexural rigidity of the wall, here E is the modulus of elasticity, t_w

is the wall thickness and v is the Poisson's ratio.

Similar approach has been used to derive the mass and stiffness properties of the diaphragms. Please see Appendix A for additional information.

The following analyses have been performed using the MDOF simplified model:

- Response spectrum analysis have been employed to compare the resulting out-of-plane wall anchorage forces with the recommended design level forces both at the roof and gallery levels of the building; the effects of out-of-plane wall stiffness and end connections have been investigated;
- Frequency response transfer functions for the displacements at all degrees of freedom have been derived;
- Study of the effect of the relative roof and gallery level diaphragm stiffnesses using response spectrum analysis.

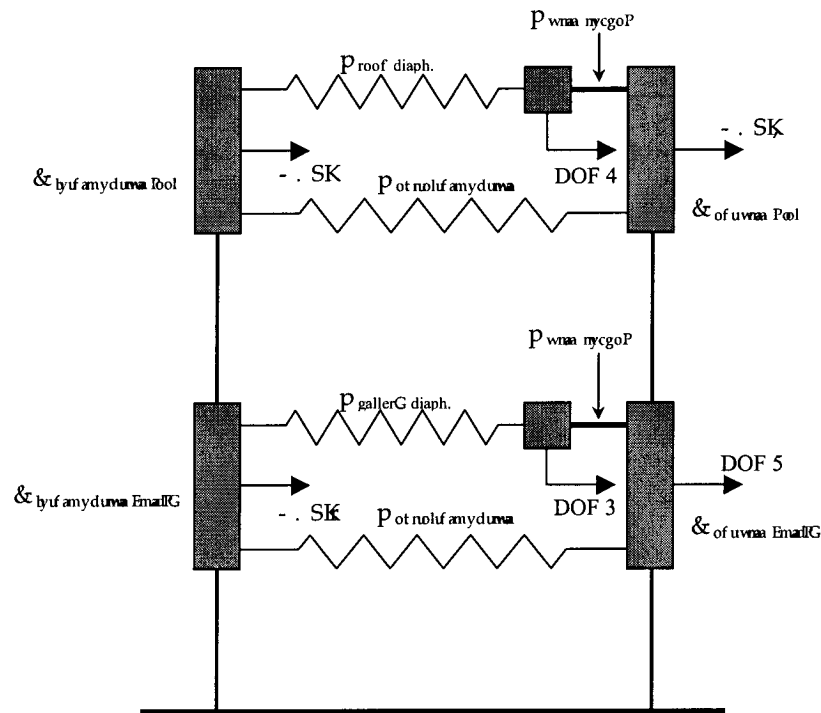


Figure 2-1. Simplified spring model.

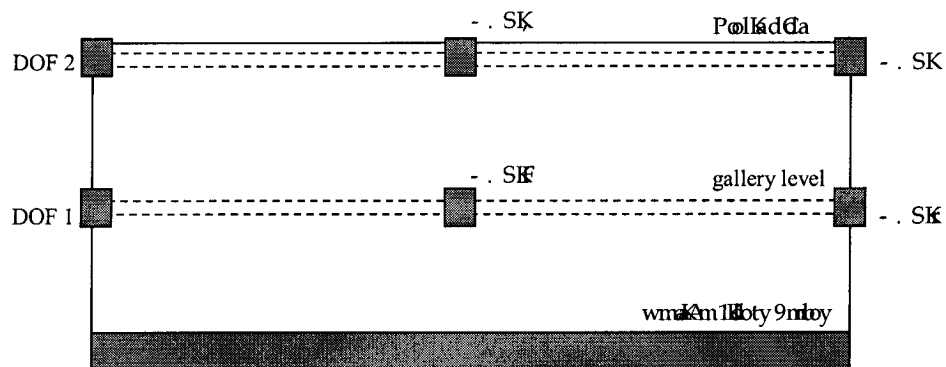


Figure 2-2. Model of the out-of-plane wall.

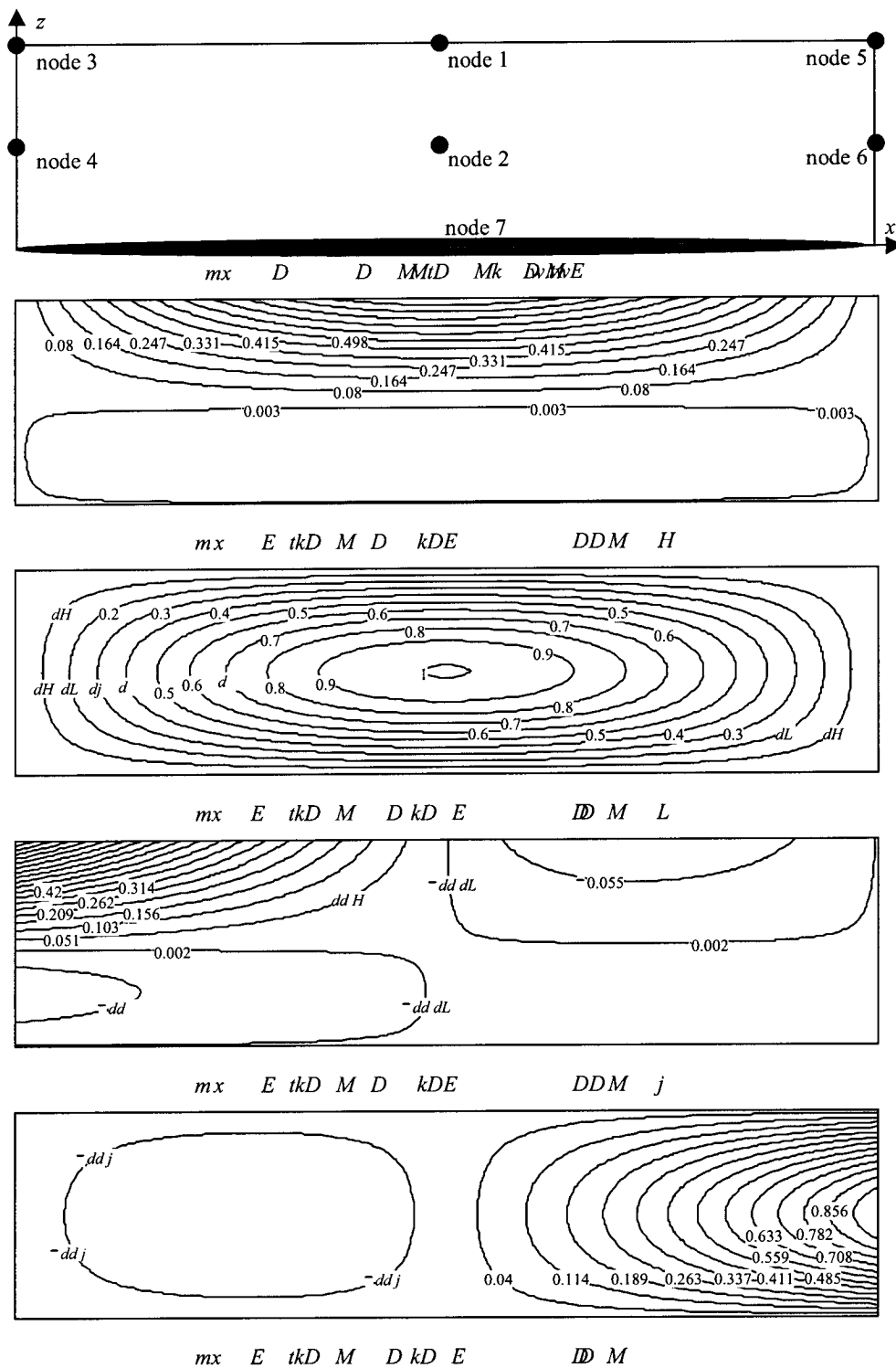


Figure 2-3. Contour plots of selected shape functions of the out-of-plane wall.

3.0 DESCRIPTION OF SUB-STATION K STRUCTURE

The example structure under consideration is the Station K power plant in PG&E's San Francisco District. The building is approximately 93 ft by 70 ft in plane and 34 ft tall built circa 1923. It has six inch perimeter reinforced concrete shear wall and interior steel framing. The perimeter walls are solid on the east and west sides of the building and have large door openings in the north and south sides of the building. This study is limited to examining the east and west walls only. The building has a gallery level at 16 ft elevation that runs along the east and west walls and is approximately 6 inches thick and 18 ft wide. The roof has is 3 to 4 inches thick and has six 7 ft by 16 ft skylights evenly distributed on east and west sides of the building and has a one large 15 ft by 78 ft skylight in the center. Both gallery and roof are supported by the concrete shear wall on the perimeter of the building in addition to the steel framing both on the perimeter and within the building. Figures 3-1 thru 3-4 show building plans, sections and elevations.

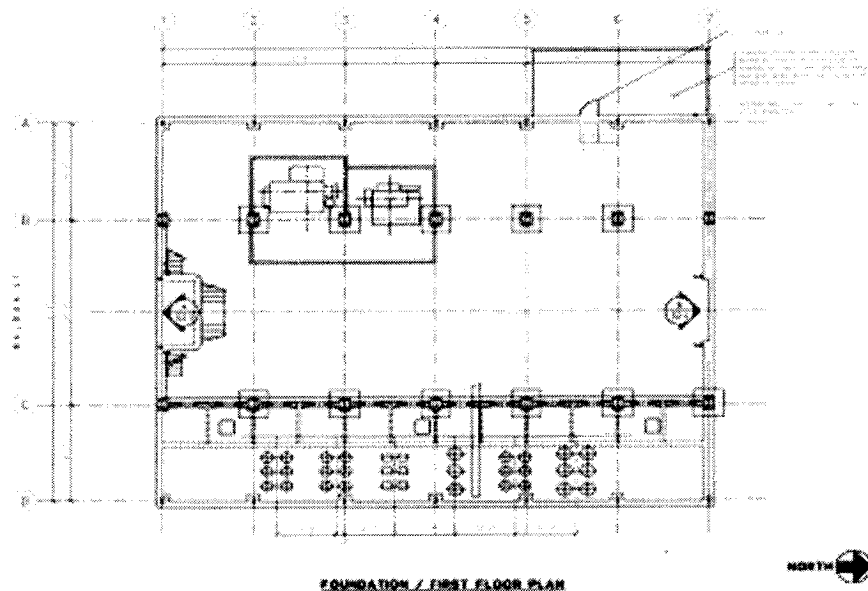


Figure 3-1: First Floor Plan

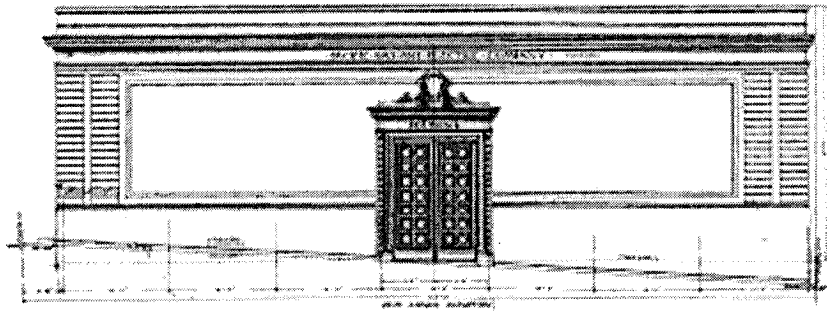


Figure 3-2: North Elevation of the Building

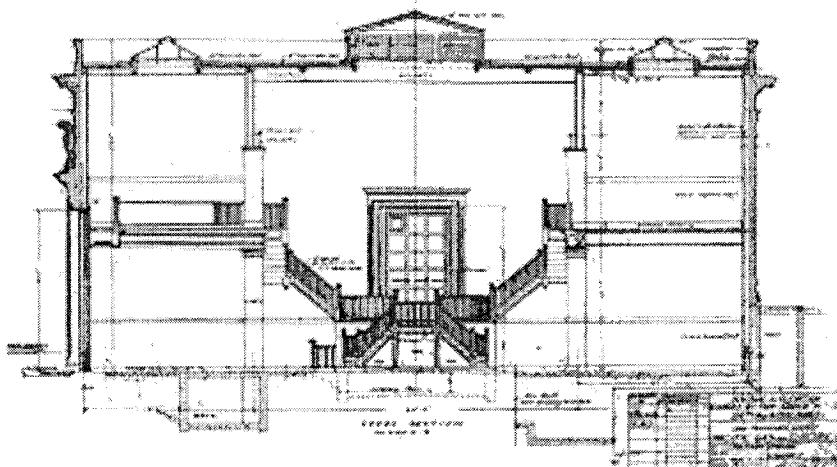


Figure 3-3: Transverse Cross Section

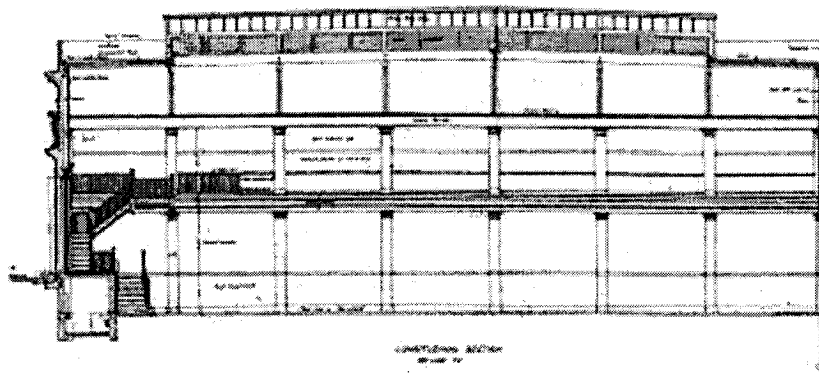


Figure 3-4: Longitudinal Cross Section

4.0 SUMMARY OF THE RESULTS

4.1 3-D Linear Dynamic Analysis (SAP2000)

For the PG&E Station K structure, the out-of-plane wall anchorage forces for the east wall on the building both at the gallery level and at the roof level have been calculated for the response spectrum shown in Figure 4-1. Assuming that out-of plane wall is connected to the diaphragms and in-plane walls with pinned connections the resulting anchorage forces are plotted in Figure 4-2, along with the design level forces recommended by FEMA 356. Note that the distribution of the anchorage forces along the wall is not uniform and the FEMA 356 recommendations for design are conservative. The maximum computed anchorage force at the gallery and roof levels are about 26% and 29% less than the design force for flexible diaphragms and rigid diaphragms, respectively.

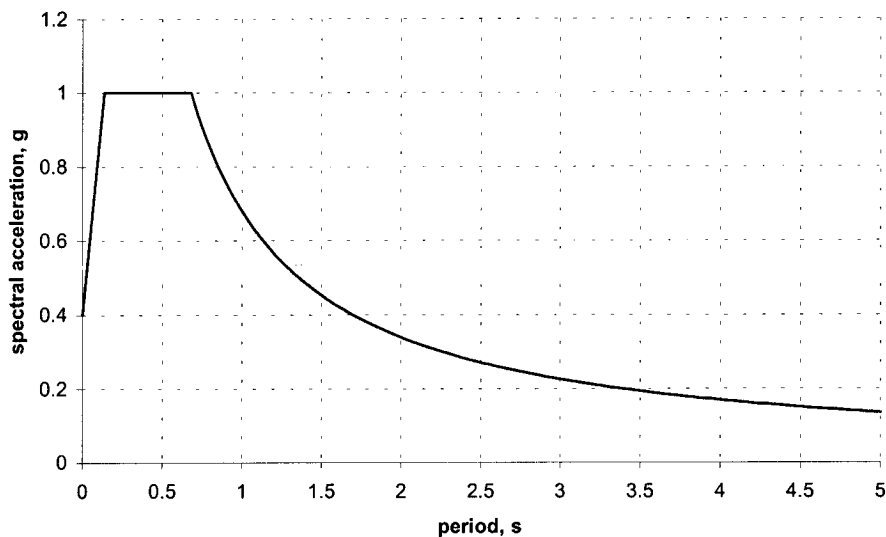


Figure 4-1. Unit Acceleration response spectrum (PGA 1.0g)

4.1.1 Effect of wall to diaphragm connection

To study the influence of the assumption of the end connections of the out-of-plane wall to the diaphragms and the in plane walls the anchorage forces are computed along the east wall of the building for two cases, i. e., pinned end conditions and fixed end conditions. Figure 4-2 shows the wall anchorage forces for the pin condition case, whereas, the Figure 4-3 shows the wall anchorage forces for the fixed end case. Also

shown in the figures are the FEMA 356 design force levels for rigid and flexible diaphragm cases. Comparing the Figures 4-2 and 4-3, for the gallery level, it is shown for the fixed end condition, that the maximum force (approximately 1000 plf) is reduced by approximately 10% as compared to the pinned end case (approximately 900 plf). However, for the roof level, the effect of rigid connection is reversed as the wall anchorage forces are increased by about 25%, (fixed end force = 400plf, pinned end force = 325 plf). The forces at the roof level are increased due to larger roof diaphragm stiffness as compared to the gallery floor diaphragm stiffness. It is expected that for the case of equal diaphragm stiffness at the two levels, the change in anchorage forces due to end conditions will be insignificant.

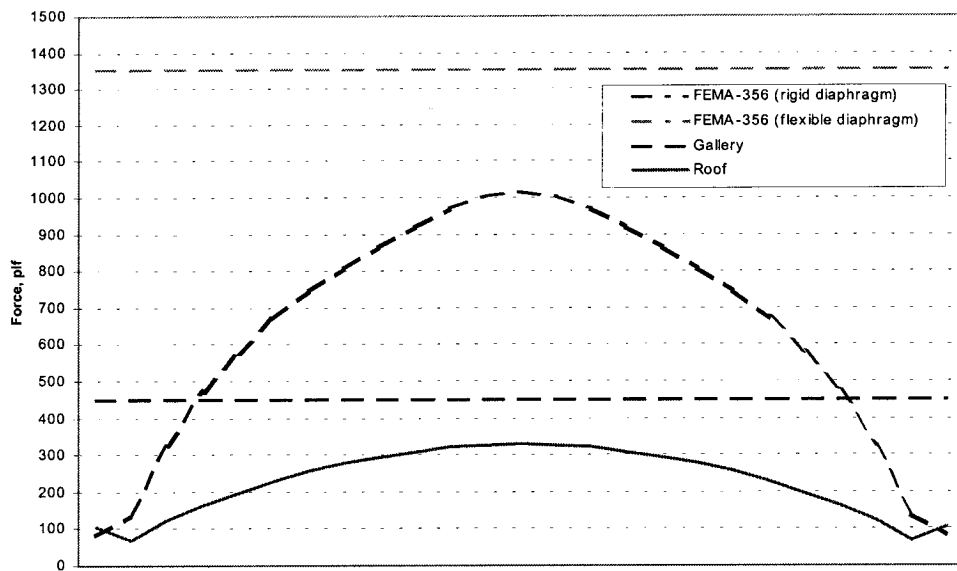
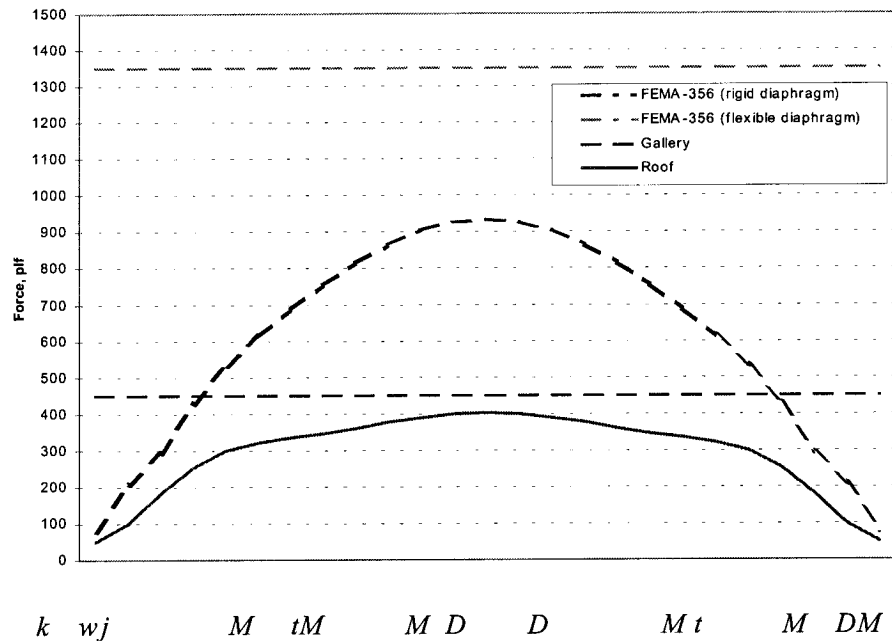


Figure 4-2. Wall anchorage forces along the east wall in case of pin connections.



4.1.2 Effect of out-of-plane stiffness

The effect of the out-of-plane wall stiffness is studied by varying the wall out of plane stiffness from zero to a value computed based upon its geometric properties. For the purposes of this parametric study, the wall end conditions were assumed to be pinned. It should be noted that FEMA 356 methodology ignores out of plane stiffness of walls in the anchorage force computation. Figure 4-4 shows the comparison of the out-of-plane wall anchorage forces at the roof and gallery levels for the two cases of wall stiffness. Also shown in the figure is the FEMA 356 force levels for rigid and flexible diaphragms. The Figure shows that the out of plane wall forces approach FEMA 356 force levels at the mid length of the wall for the case of zero wall stiffness. In addition, the Figure shows that for the gallery level, the maximum wall anchorage force is approximately 1300 plf (maximum level) when wall stiffness is ignored (per FEMA 356). Whereas, this force is reduced to approximately 1000 plf when wall's stiffness is accounted for. This represents a 30% reduction in wall anchorage forces. Similarly, the wall anchorage forces are reduced by approximately 20% at the roof level. Therefore, by including the wall out of plane stiffness, significant reduction in wall out of plane forces can be achieved.

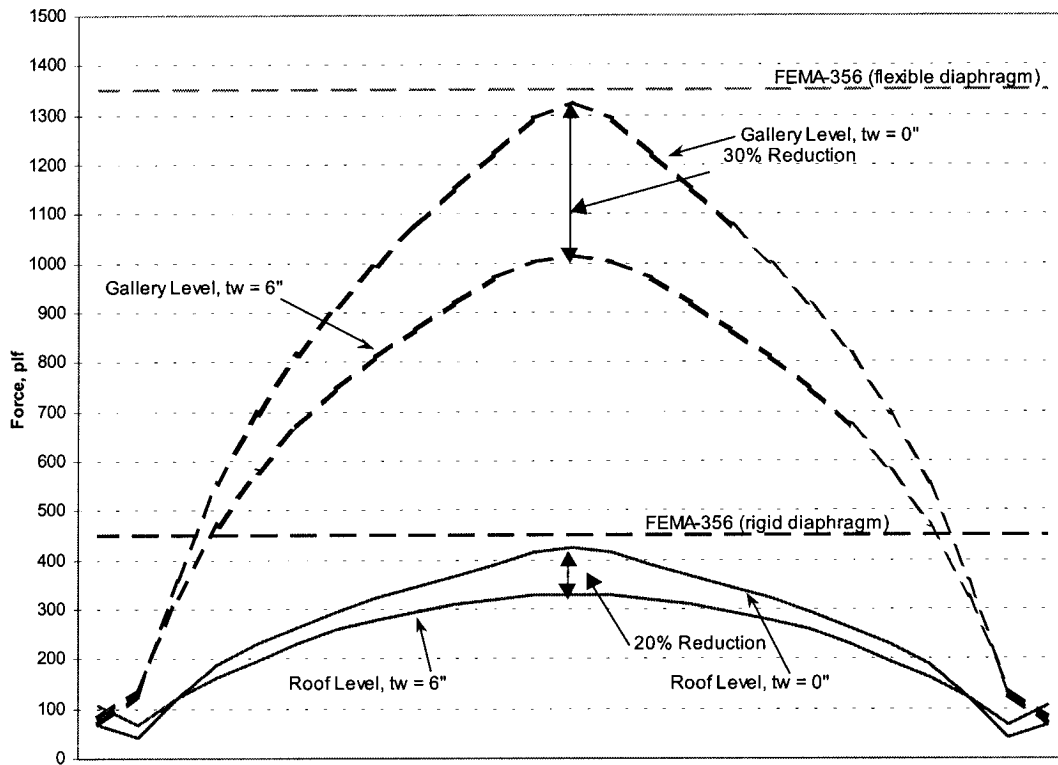


Figure 4-4. Effect of out-of-plane wall stiffness on anchorage forces.

4.1.3 Effect of out-of-plane rocking

Time history analyses have been performed to study the effect of rocking of the out-of-plane wall on the wall anchorage forces. Non-linear spring-damper elements have been used to model the out-of-plane wall rocking at the foundation. The Northridge 1994 earthquake record have been scaled to match the spectral acceleration of the response spectra shown in Figure 4-1 at the predominant vibration period of the out-of-plane wall. The scaled response spectrum of Northridge earthquake together with the design response spectrum used in previous analyses are plotted in Figure 4-5.

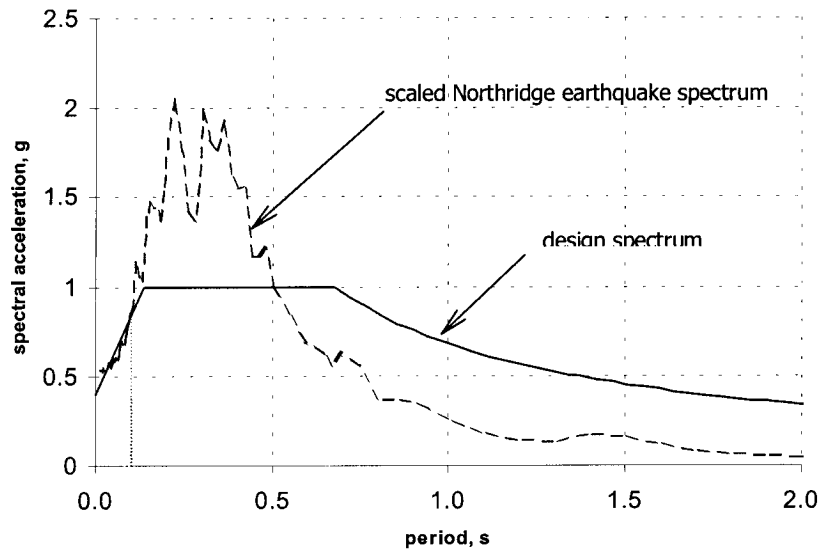


Figure 4-5. Acceleration response spectra.

The results of the analyses are plotted in Figures 4-6 and 4-7 for the cases of low, 5%, and high, 40% damping at the foundation of the out-of-plane wall. Comparing the results in Figures 4-6 and 4-7 one can see that the out-of-plane wall anchorage forces are not sensitive to the assumed damping at the foundation for the rocking of the out-of plane wall. Note that the results of the time history analyses shown in Figures 4-6 and 4-7 are in good agreement with the results of the response spectrum analysis shown in Figure 4-2. This demonstrates that the effect of rocking of the out-of-plane wall on wall anchorage forces is negligible.

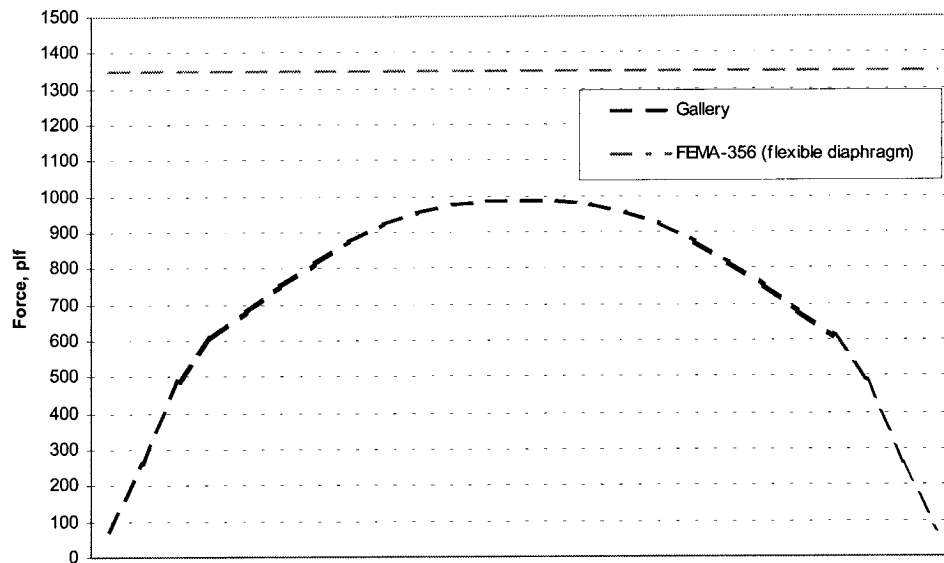


Figure 4-6. Wall anchorage forces at the gallery level along the east wall accounting for the out-of-plane wall rocking, assuming 5% damping at the foundation.

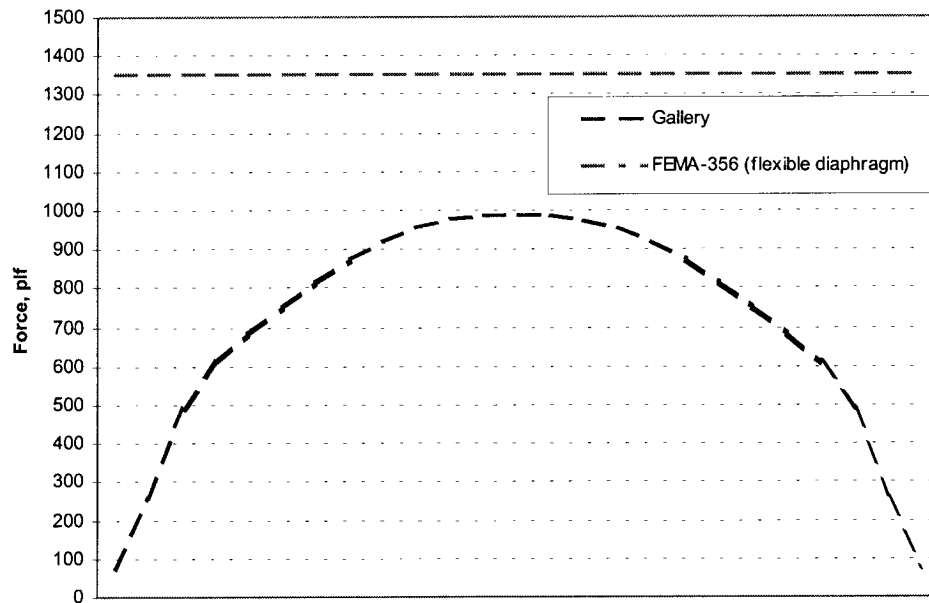


Figure 4-7. Wall anchorage forces at the gallery level along the east wall accounting for the out-of-plane wall rocking, assuming 40% damping at the foundation.

4.2 MDOF Simplified Spring Model

The simplified model is developed as means of developing simplified procedures for computing the wall anchorage forces for design purposes. This model allows performing sensitivity analysis on various parameters of the structure that could affect the response of the building and the magnitude of the wall anchorage forces.

Comparison of the periods and mode shapes of the simplified six degree of freedom model showed a good match with those of the detailed SAP2000 model developed as part of this study.

Figure 4-8 shows the amplification functions for the displacements at the ends of the out-of-plane wall and at the center of the out-of-plane wall, both at the roof as well as gallery levels of the building.

The displacement responses at the ends of the out-of-plane wall are much smaller than those at the center of the wall and the contribution to these responses is coming mainly from the second and third modes of vibration of the building. Whereas the main contributions to the responses at the center of the wall at the roof and at the gallery levels is coming from the second and first modes, respectively. Also note that there is considerable contribution from the second and third modes of vibration to the displacement responses at the gallery and roof levels, respectively.

Figures 4-9 and 4-10 demonstrate the sensitivity of the responses to the changes in the stiffness of the gallery diaphragm and the roof diaphragm, respectively. Examining the displacement responses of the building in Figure 4-9 one can see that the response of the building is dominated by the second mode of the building that is the bulging of the out-of-plane wall until the gallery stiffness is as high as 3500 kips /in. Similarly Figure 4-10 demonstrates that the response of the building is dominated by the first mode of vibration, i.e. larger displacement at the roof level than at the gallery level, until the roof stiffness reaches 900 kips/in.

A close match can be observed between the wall anchorage forces at the vertical lines shown in Figures 4-9 and 4-10 when compared with the ones in Figure 4-2. This once again demonstrates the ability of the simplified model to capture the response of the building with acceptable accuracy.

In both cases the anchorage force at the gallery level is more sensitive to the changes in the diaphragm stiffness than the one at the roof level.

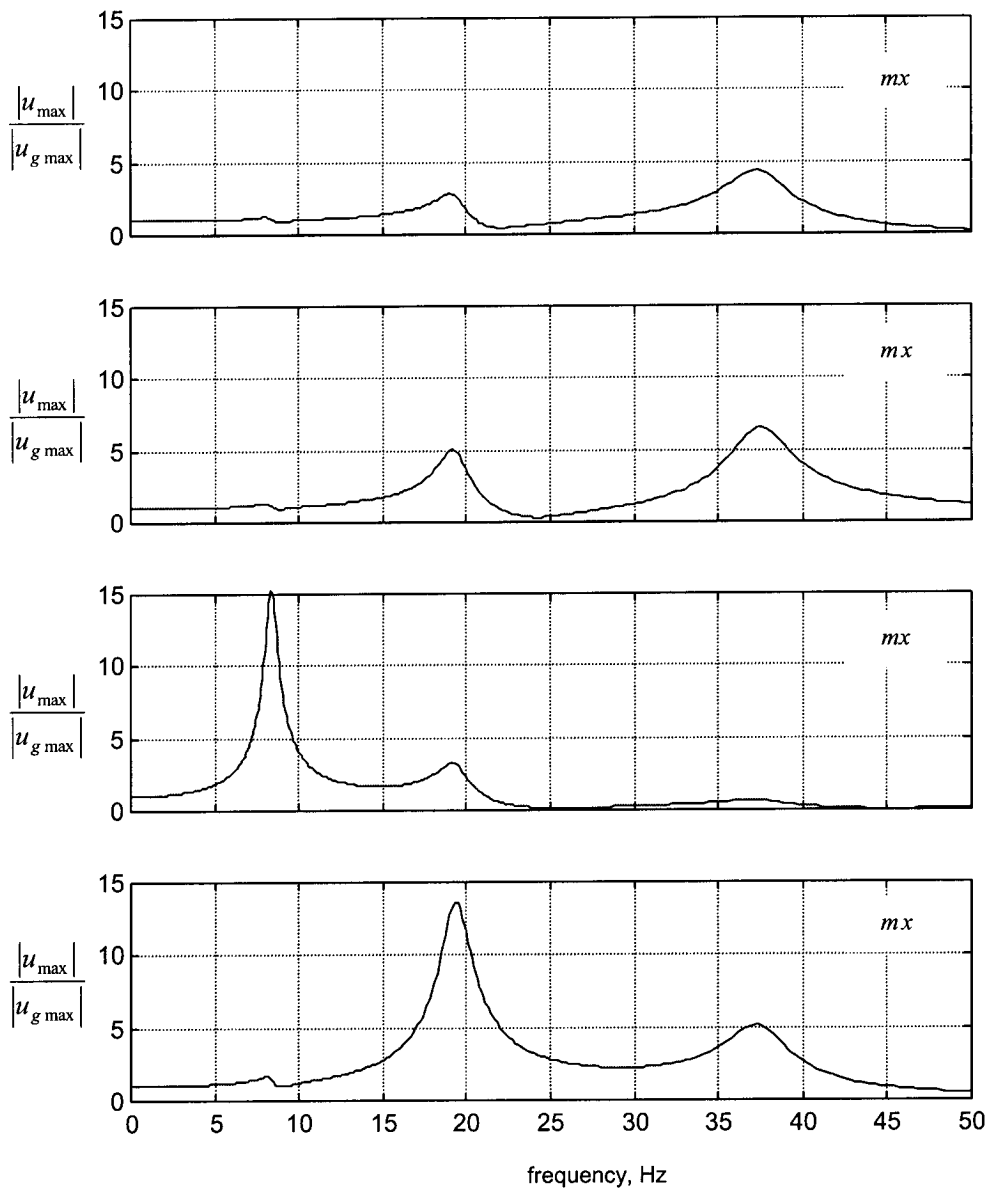


Figure 4-8. Amplification functions for the displacements (a) at the ends of the out-of-plane wall at gallery level, (b) at the ends of the out-of-plane wall at roof level, (c) at the center of the out-of-plane wall at gallery level, and (d) at the center of the out-of-plane wall at roof level

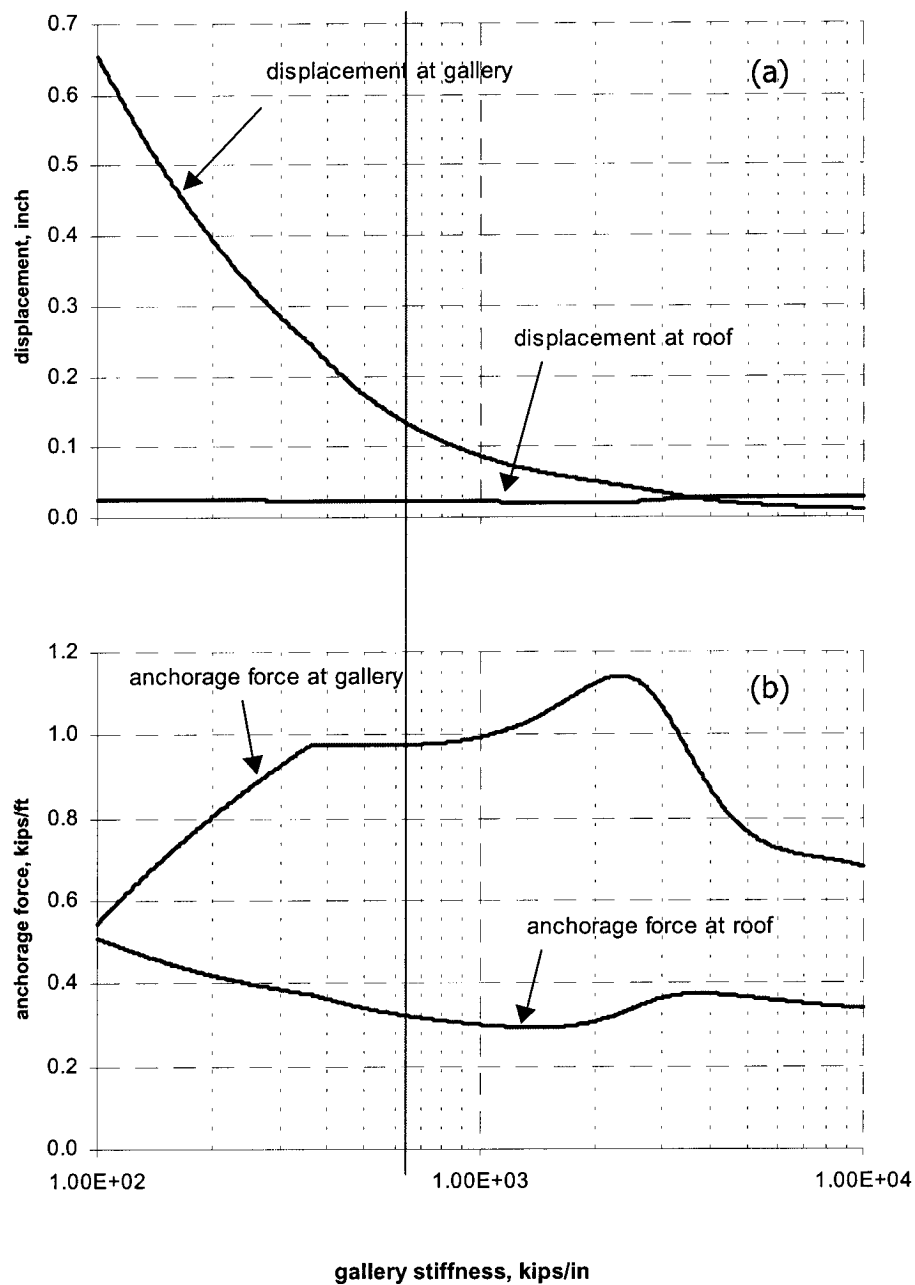


Figure 4-9. Sensitivity of the (a) displacements and (b) wall anchorage forces to the change in gallery stiffness. The solid vertical line indicates the actual equivalent stiffness of the gallery diaphragm stiffness.

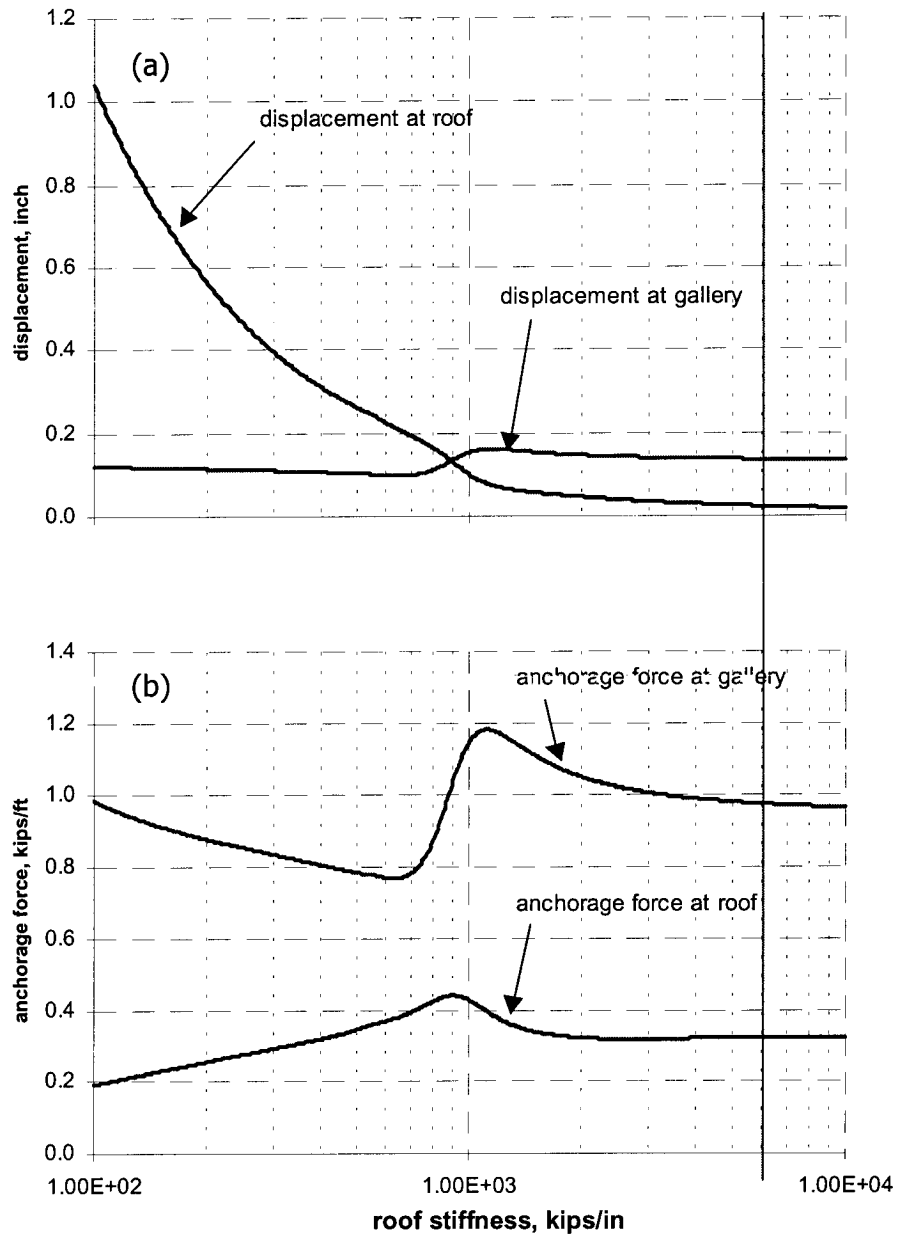


Figure 4-10. Sensitivity of the (a) displacements and (b) wall anchorage forces to the change in roof stiffness. The solid vertical line indicates the actual equivalent stiffness of the roof diaphragm stiffness.

5.0 RECOMMENDATIONS

In this study it has been demonstrated that in all cases for the building under consideration the recommended FEMA 356 design forces for the wall anchorage design were always conservative. It was shown that the distribution of the anchorage forces along the length of the wall is not uniform and decreases significantly as it moves away from the center of the wall.

This study has also shown that the following parameters have significant influence on the wall anchorage forces:

- Response reduction factor (in comparison to the FEMA 356 factor)
- Out of plane wall stiffness
- Location of wall anchor force relative to overall length of the wall.
- Diaphragm stiffness and the relative stiffnesses of both the roof and the gallery floor diaphragm

This study can be extended to propose design equations based upon the analyses presented in this report. The proposed design equation may take the following form:

$$F_p = \chi S_{xs} R_p L_x t_w t_{d1} t_{d2}$$

Where:

χ = Factor from FEMA356 Table 2 - 4

S_{xs} = Spectral Response Acceleration Short Period (FEMA356)

R_p = Response Reduction Parameter

L_x = Wall anchor location Parameter

t_w = Wall out of plane stiffness Parameter

t_d = Diaphragm stiffness Parameter

6.0 REFERENCES

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2. Federal Emergency Management Agency, (2000) **"Prestandard and Commentary for the Seismic Rehabilitation of Buildings"**, (FEMA 356)
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5. Clough R.W. and J. Penzien, (1993) **"Dynamics of Structures"**, Second Edition, McGraw-Hill, Inc., New York.
6. Timoshenko S., (1959) **"Theory of Plates and Shells"**, Reissue, McGraw-Hill Companies, New York.

Stiffness and Mass Matrices for the Out-of-plane Wall

Wall Length $L := 93 \text{ ft}$

Modulus of Elasticity $E := 3600 \text{ ksi}$

Wall Thickness $t_w := 6 \text{ in}$

Poisson's Ratio $\nu := 0.2$

First Floor Height $h_1 := 16 \text{ ft}$

Shear Modulus $G := \frac{E}{2 \cdot (1 + \nu)}$

Gallery Level Height $h_2 := 18 \text{ ft}$

ONE DIMENSIONAL SHAPE FUNCTIONS ALONG VERTICAL Z AXIS

$$a_{1z} := \frac{-1}{h_2 \cdot (h_1^3 + 4 \cdot h_1^2 \cdot h_2 + 4 \cdot h_1 \cdot h_2^2 + h_2^3)}$$

$$b_{1z} := -2 \cdot (h_1 + h_2) \cdot a_{1z}$$

$$d_{1z} := a_{1z} \cdot h_1^2 \cdot (h_1 + 2 \cdot h_2)$$

$$\psi_{1z}(z) := (a_{1z} \cdot z^4 + b_{1z} \cdot z^3 + d_{1z} \cdot z)$$

$$a_{2z} := \frac{1}{(h_1^2 + 3 \cdot h_1 \cdot h_2 + h_2^2) \cdot (h_1 \cdot h_2)}$$

$$b_{2z} := -2 \cdot (h_1 + h_2) \cdot a_{2z}$$

$$d_{2z} := \frac{(a_{2z} \cdot h_1^4 + 2 \cdot a_{2z} \cdot h_1^3 \cdot h_2 + 1)}{h_1}$$

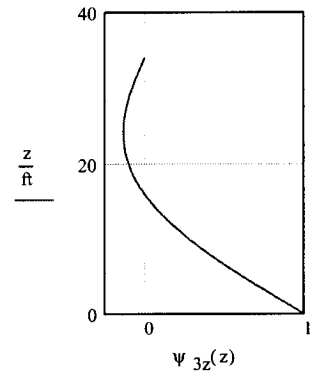
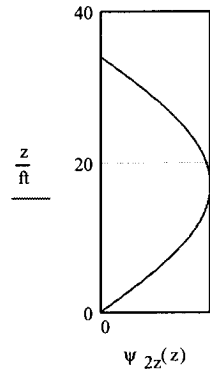
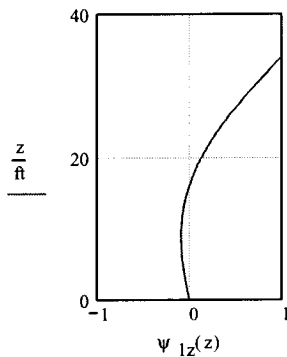
$$\psi_{2z}(z) := (a_{2z} \cdot z^4 + b_{2z} \cdot z^3 + d_{2z} \cdot z)$$

$$a_{3z} := \frac{-1}{h_1 \cdot (h_1^3 + 4 \cdot h_1^2 \cdot h_2 + 4 \cdot h_1 \cdot h_2^2 + h_2^3)}$$

$$b_{3z} := -2 \cdot (h_1 + h_2) \cdot a_{3z}$$

$$d_{3z} := \frac{(a_{3z} \cdot h_1^4 + 2 \cdot a_{3z} \cdot h_1^3 \cdot h_2 - 1)}{h_1}$$

$$\psi_{3z}(z) := a_{3z} \cdot z^4 + b_{3z} \cdot z^3 + d_{3z} \cdot z + 1$$



ONE DIMENSIONAL SHAPE FUNCTIONS ALONG HORIZONTAL X AXIS

$$a_{1x} := \frac{-8}{(5 \cdot L^4)} \quad b_{1x} := \frac{16}{(5 \cdot L^3)} \quad d_{1x} := \frac{-13}{(5 \cdot L)}$$

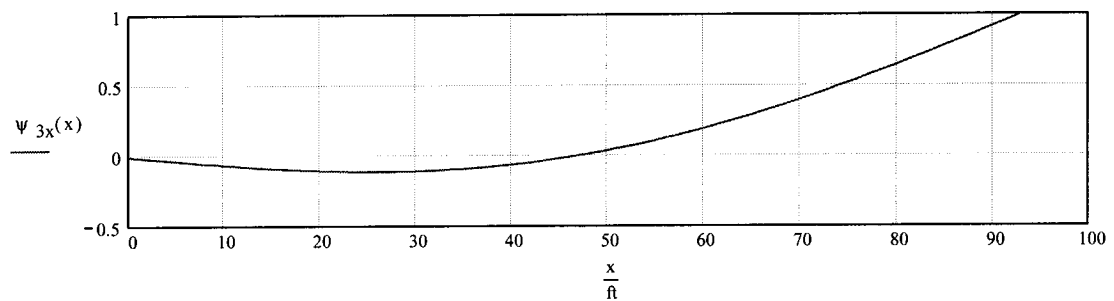
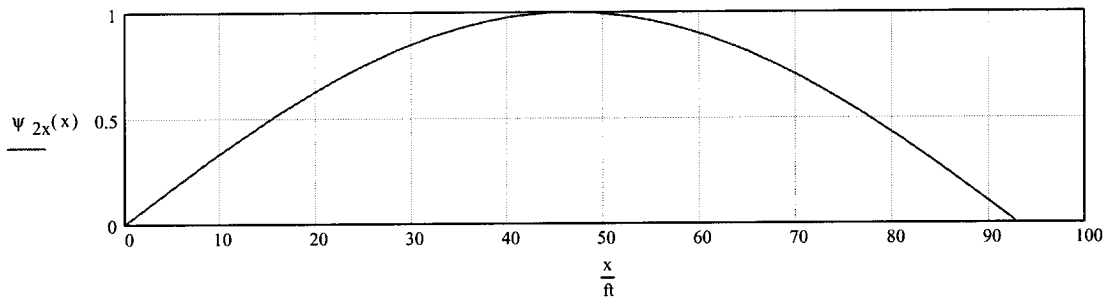
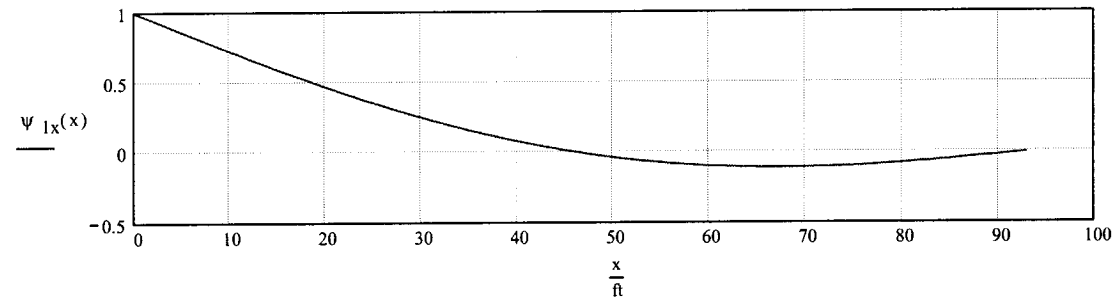
$$\psi_{1x}(x) := a_{1x} \cdot x^4 + b_{1x} \cdot x^3 + d_{1x} \cdot x + 1$$

$$a_{2x} := \frac{16}{(5 \cdot L^4)} \quad b_{2x} := \frac{-32}{(5 \cdot L^3)} \quad d_{2x} := \frac{16}{(5 \cdot L)}$$

$$\psi_{2x}(x) := a_{2x} \cdot x^4 + b_{2x} \cdot x^3 + d_{2x} \cdot x$$

$$a_{3x} := \frac{-8}{(5 \cdot L^4)} \quad b_{3x} := \frac{16}{(5 \cdot L^3)} \quad d_{3x} := \frac{-3}{(5 \cdot L)}$$

$$\psi_{3x}(x) := a_{3x} \cdot x^4 + b_{3x} \cdot x^3 + d_{3x} \cdot x$$



TWO DIMENSIONAL SHAPE FUNCTIONS OF THE WALL

$$\psi_{11}(x, z) := \psi_{1z}(z) \cdot \psi_{1x}(x)$$

$$\psi_{12}(x, z) := \psi_{1z}(z) \cdot \psi_{2x}(x)$$

$$\psi_{13}(x, z) := \psi_{1z}(z) \cdot \psi_{3x}(x)$$

$$\psi_{21}(x, z) := \psi_{2z}(z) \cdot \psi_{1x}(x)$$

$$\psi_{22}(x, z) := \psi_{2z}(z) \cdot \psi_{2x}(x)$$

$$\psi_{23}(x, z) := \psi_{2z}(z) \cdot \psi_{3x}(x)$$

$$\psi_7(x, z) := \psi_{3z}(z)$$

STIFFNESS MATRIX OF THE OUT-OF-PLANE WALL

Compute the elements of the stiffness matrices using the above mode shapes and definitions given in section 2.2.

Flexural Rigidity of the Plate Bending: $D := \frac{E \cdot t_w^3}{12 \cdot (1 - \nu)}$

Bending moments as defined in section 2.2

$$M_{z11}(x, z) := -D \cdot \left(\frac{d^2}{dx^2} \psi_{11}(xz) + \nu \frac{d^2}{dz^2} \psi_{11}(x, z) \right)$$

$$M_{x11}(x, z) := -D \cdot \left(\nu \frac{d^2}{dx^2} \psi_{11}(x, z) + \frac{d^2}{dz^2} \psi_{11}(x, z) \right)$$

$$M_{xz11}(x, z) := D \cdot (1 - \nu) \frac{d}{dx} \frac{d}{dz} \psi_{11}(x, z)$$

$$M_{z12}(x, z) := -D \cdot \left(\frac{d^2}{dx^2} \psi_{12}(xz) + \nu \frac{d^2}{dz^2} \psi_{12}(x, z) \right)$$

$$M_{x12}(x, z) := -D \cdot \left(\nu \frac{d^2}{dx^2} \psi_{12}(x, z) + \frac{d^2}{dz^2} \psi_{12}(x, z) \right)$$

$$M_{xz12}(x, z) := D \cdot (1 - \nu) \frac{d}{dx} \frac{d}{dz} \psi_{12}(x, z)$$

$$M_{z13}(x, z) := -D \cdot \left(\frac{d^2}{dx^2} \psi_{13}(xz) + \nu \frac{d^2}{dz^2} \psi_{13}(x, z) \right)$$

$$M_{x13}(x, z) := -D \cdot \left(\nu \frac{d^2}{dx^2} \psi_{13}(x, z) + \frac{d^2}{dz^2} \psi_{13}(x, z) \right)$$

$$M_{xz13}(x, z) := D \cdot (1 - \nu) \frac{d}{dx} \frac{d}{dz} \psi_{13}(x, z)$$

$$M_{z21}(x,z):=-D\cdot\left(\frac{d^2}{dx^2}\psi_{21}(xz)+v\frac{d^2}{dz^2}\psi_{21}(x,z)\right)$$

$$M_{x21}(x,z):=-D\cdot\left(v\frac{d^2}{dx^2}\psi_{21}(x,z)+\frac{d^2}{dz^2}\psi_{21}(x,z)\right)$$

$$M_{xz21}(x,z):=D\cdot(1-v)\frac{d}{dx}\frac{d}{dz}\psi_{21}(x,z)$$

$$M_{z22}(x,z):=-D\cdot\left(\frac{d^2}{dx^2}\psi_{22}(xz)+v\frac{d^2}{dz^2}\psi_{22}(x,z)\right)$$

$$M_{x22}(x,z):=-D\cdot\left(v\frac{d^2}{dx^2}\psi_{22}(x,z)+\frac{d^2}{dz^2}\psi_{22}(x,z)\right)$$

$$M_{xz22}(x,z):=D\cdot(1-v)\frac{d}{dx}\frac{d}{dz}\psi_{22}(x,z)$$

$$M_{z23}(x,z):=-D\cdot\left(\frac{d^2}{dx^2}\psi_{23}(xz)+v\frac{d^2}{dz^2}\psi_{23}(x,z)\right)$$

$$M_{x23}(x,z):=-D\cdot\left(v\frac{d^2}{dx^2}\psi_{23}(x,z)+\frac{d^2}{dz^2}\psi_{23}(x,z)\right)$$

$$M_{xz23}(x,z):=D\cdot(1-v)\frac{d}{dx}\frac{d}{dz}\psi_{23}(x,z)$$

$$M_{z\gamma}(x,z):=-D\cdot\left[v\cdot\left(\frac{d^2}{dz^2}\psi_{\gamma}(x,z)\right)\right]$$

$$M_{x\gamma}(x,z):=-D\cdot\left(\frac{d^2}{dz^2}\psi_{\gamma}(x,z)\right)$$

$$M_{xz\gamma}(x,z):=D\cdot(1-v)\frac{d}{dx}\frac{d}{dz}\psi_{\gamma}(x,z)$$

Compute the elements of the stiffness matrix

$$k_{1,1} := \int_0^L \int_0^{(h_1+h_2)} \left[M_{x12}(x,z) \cdot \left(-\frac{d^2}{dz^2} \psi_{12}(x,z) \right) + M_{z12}(x,z) \cdot \left(-\frac{d^2}{dx^2} \psi_{12}(xz) \right) + 2 \cdot M_{xz12}(x,z) \cdot \left(\frac{d}{dx} \frac{d}{dz} \psi_{12}(x,z) \right) \right] dz dx$$

$$k_{1,2} := \int_0^L \int_0^{(h_1+h_2)} \left[M_{x12}(x,z) \cdot \left(-\frac{d^2}{dz^2} \psi_{22}(x,z) \right) + M_{z12}(x,z) \cdot \left(-\frac{d^2}{dx^2} \psi_{22}(xz) \right) + 2 \cdot M_{xz12}(x,z) \cdot \left(\frac{d}{dx} \frac{d}{dz} \psi_{22}(x,z) \right) \right] dz dx$$

$$k_{1,3} := \int_0^L \int_0^{(h_1+h_2)} \left[M_{x12}(x,z) \cdot \left(-\frac{d^2}{dz^2} \psi_{11}(x,z) \right) + M_{z12}(x,z) \cdot \left(-\frac{d^2}{dx^2} \psi_{11}(xz) \right) + 2 \cdot M_{xz12}(x,z) \cdot \left(\frac{d}{dx} \frac{d}{dz} \psi_{11}(x,z) \right) \right] dz dx$$

$$k_{1,4} := \int_0^L \int_0^{(h_1+h_2)} \left[M_{x12}(x,z) \cdot \left(-\frac{d^2}{dz^2} \psi_{21}(x,z) \right) + M_{z12}(xz) \cdot \left(-\frac{d^2}{dx^2} \psi_{21}(xz) \right) + 2 \cdot M_{xz12}(x,z) \cdot \left(\frac{d}{dx} \frac{d}{dz} \psi_{21}(x,z) \right) \right] dz dx$$

$$k_{1,5} := k_{1,3}$$

$$k_{1,6} := k_{1,4}$$

$$k_{1,7} := \int_0^L \int_0^{(h_1+h_2)} \left(M_{x12}(x,z) - \frac{d^2}{dz^2} \psi_7(x,z) \right) dz dx$$

$$k_{2,2} := \int_0^L \int_0^{(h_1+h_2)} \left[M_{xz22}(x,z) \cdot \left(-\frac{d^2}{dz^2} \psi_{22}(x,z) \right) + M_{z22}(x,z) \cdot \left(-\frac{d^2}{dx^2} \psi_{22}(xz) \right) + 2 \cdot M_{xz22}(x,z) \cdot \left(\frac{d}{dx} \frac{d}{dz} \psi_{22}(x,z) \right) \right] dz dx$$

$$k_{2,3} := \int_0^L \int_0^{(h_1+h_2)} \left[M_{xz22}(x,z) \cdot \left(-\frac{d^2}{dz^2} \psi_{11}(x,z) \right) + M_{z22}(x,z) \cdot \left(-\frac{d^2}{dx^2} \psi_{11}(xz) \right) + 2 \cdot M_{xz22}(x,z) \cdot \left(\frac{d}{dx} \frac{d}{dz} \psi_{11}(x,z) \right) \right] dz dx$$

$$k_{2,4} := \int_0^L \int_0^{(h_1+h_2)} \left[M_{xz22}(x,z) \cdot \left(-\frac{d^2}{dz^2} \psi_{21}(x,z) \right) + M_{z22}(xz) \cdot \left(-\frac{d^2}{dx^2} \psi_{21}(xz) \right) + 2 \cdot M_{xz22}(x,z) \cdot \left(\frac{d}{dx} \frac{d}{dz} \psi_{21}(x,z) \right) \right] dz dx$$

$$k_{2,5} := k_{2,3}$$

$$k_{2,6} := k_{2,4}$$

$$k_{2,7} := \int_0^L \int_0^{(h_1+h_2)} \left(M_{xz22}(x,z) - \frac{d^2}{dz^2} \psi_{77}(x,z) \right) dz dx$$

$$k_{3,3} := \int_0^L \int_0^{(h_1+h_2)} \left[M_{xz11}(x,z) \cdot \left(-\frac{d^2}{dz^2} \psi_{11}(x,z) \right) + M_{z11}(x,z) \cdot \left(-\frac{d^2}{dx^2} \psi_{11}(xz) \right) + 2 \cdot M_{xz11}(x,z) \cdot \left(\frac{d}{dx} \frac{d}{dz} \psi_{11}(x,z) \right) \right] dz dx$$

$$k_{3,4} := \int_0^L \int_0^{(h_1+h_2)} \left[M_{xz11}(x,z) \cdot \left(-\frac{d^2}{dz^2} \psi_{21}(x,z) \right) + M_{z11}(xz) \cdot \left(-\frac{d^2}{dx^2} \psi_{21}(xz) \right) + 2 \cdot M_{xz11}(x,z) \cdot \left(\frac{d}{dx} \frac{d}{dz} \psi_{21}(x,z) \right) \right] dz dx$$

$$k_{3,5} := \int_0^L \int_0^{(h_1+h_2)} \left[M_{x11}(x,z) \cdot \left(-\frac{d^2}{dz^2} \psi_{13}(x,z) \right) + M_{z11}(xz) \cdot \left(-\frac{d^2}{dx^2} \psi_{13}(xz) \right) + 2 \cdot M_{xz11}(x,z) \cdot \left(\frac{d}{dx} \frac{d}{dz} \psi_{13}(x,z) \right) \right] dz dx$$

$$k_{3,6} := \int_0^L \int_0^{(h_1+h_2)} \left[M_{x11}(x,z) \cdot \left(-\frac{d^2}{dz^2} \psi_{23}(x,z) \right) + M_{z11}(xz) \cdot \left(-\frac{d^2}{dx^2} \psi_{23}(xz) \right) + 2 \cdot M_{xz11}(x,z) \cdot \left(\frac{d}{dx} \frac{d}{dz} \psi_{23}(x,z) \right) \right] dz dx$$

$$k_{3,7} := \int_0^L \int_0^{(h_1+h_2)} \left(M_{x11}(x,z) - \frac{d^2}{dz^2} \psi_7(x,z) \right) dz dx$$

$$k_{4,4} := \int_0^L \int_0^{(h_1+h_2)} \left[M_{x21}(x,z) \cdot \left(-\frac{d^2}{dz^2} \psi_{21}(x,z) \right) + M_{z21}(xz) \cdot \left(-\frac{d^2}{dx^2} \psi_{21}(xz) \right) + 2 \cdot M_{xz21}(x,z) \cdot \left(\frac{d}{dx} \frac{d}{dz} \psi_{21}(x,z) \right) \right] dz dx$$

$$k_{4,5} := \int_0^L \int_0^{(h_1+h_2)} \left[M_{x21}(x,z) \cdot \left(-\frac{d^2}{dz^2} \psi_{13}(x,z) \right) + M_{z21}(xz) \cdot \left(-\frac{d^2}{dx^2} \psi_{13}(xz) \right) + 2 \cdot M_{xz21}(x,z) \cdot \left(\frac{d}{dx} \frac{d}{dz} \psi_{13}(x,z) \right) \right] dz dx$$

$$k_{4,6} := \int_0^L \int_0^{(h_1+h_2)} \left[M_{x21}(x,z) \cdot \left(-\frac{d^2}{dz^2} \psi_{23}(x,z) \right) + M_{z21}(xz) \cdot \left(-\frac{d^2}{dx^2} \psi_{23}(xz) \right) + 2 \cdot M_{xz21}(x,z) \cdot \left(\frac{d}{dx} \frac{d}{dz} \psi_{23}(x,z) \right) \right] dz dx$$

$$k_{4,7} := \int_0^L \int_0^{(h_1+h_2)} \left(M_{x21}(x,z) - \frac{d^2}{dz^2} \psi_7(x,z) \right) dz dx$$

$$k_{5,5} := k_{3,3}$$

$$k_{5,6} := k_{3,4}$$

$$k_{5,7} := k_{3,7}$$

$$k_{6,6} := k_{4,4}$$

$$k_{6,7} := k_{4,7}$$

$$k_{7,7} := \int_0^L \int_0^{(h_1 + h_2)} \left(M_{x7}(x, z) - \frac{d^2}{dz^2} \psi_7(x, z) \right) dz dx$$

Stiffness matrix of the out-of-plane wall

$$k = \begin{bmatrix} 10.374 & -19.153 & -0.558 & -0.26 & -0.558 & -0.26 & 10.415 \\ -19.153 & 42.707 & -0.626 & 0.327 & -0.626 & 0.327 & -22.956 \\ -0.558 & -0.626 & 3.435 & -5.788 & -0.201 & 0.729 & 3.01 \\ -0.26 & 0.327 & -5.788 & 12.413 & 0.729 & -1.436 & -5.984 \\ -0.558 & -0.626 & -0.201 & 0.729 & 3.435 & -5.788 & 3.01 \\ -0.26 & 0.327 & 0.729 & -1.436 & -5.788 & 12.413 & -5.984 \\ 10.415 & -22.956 & 3.01 & -5.984 & 3.01 & -5.984 & 18.489 \end{bmatrix} \frac{\text{• kips}}{\text{in}}$$

Compute the elements of the mass matrix based on definition given in section 2.2.

$$m_{1,1} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (\tau_w \cdot 150 \text{ pcf}) \cdot \psi_{12}(x, z)^2 dz dx$$

$$m_{2,2} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (\tau_w \cdot 150 \text{ pcf}) \cdot \psi_{22}(x, z)^2 dz dx$$

$$m_{3,3} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (\tau_w \cdot 150 \text{ pcf}) \cdot \psi_{11}(x, z)^2 dz dx$$

$$m_{5,5} := m_{3,3}$$

$$m_{4,4} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (\tau_w \cdot 150 \text{ pcf}) \cdot \psi_{21}(x, z)^2 dz dx$$

$$m_{6,6} := m_{4,4}$$

$$m_{7,7} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (\tau_w \cdot 150 \text{ pcf}) \cdot \psi_7(x, z)^2 dz dx$$

$$m_{1,2} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (\tau_w \cdot 150 \text{ pcf}) \cdot \psi_{12}(x, z) \cdot \psi_{22}(x, z) dz dx$$

$$m_{1,3} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (\tau_w \cdot 150 \text{ pcf}) \cdot \psi_{12}(x, z) \cdot \psi_{11}(x, z) dz dx$$

$$m_{15} := m_{1,3}$$

$$m_{1,4} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (\tau_w \cdot 150 \text{ pcf}) \cdot \psi_{12}(x, z) \cdot \psi_{21}(x, z) dz dx$$

$$m_{16} := m_{1,4}$$

$$m_{1,7} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (\tau_w \cdot 150 \text{ pcf}) \cdot \psi_{12}(x, z) \cdot \psi_7(x, z) dz dx$$

$$\psi_{11} = \psi_{12}$$

$$m_{2,3} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (t_w \cdot 150 \text{ pcf}) \cdot \psi_{22}(x, z) \cdot \psi_{11}(x, z) dz dx$$

$$m_{25} := m_{2,3}$$

$$m_{2,4} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (t_w \cdot 150 \text{ pcf}) \cdot \psi_{22}(x, z) \cdot \psi_{21}(x, z) dz dx$$

$$m_{26} := m_{2,4}$$

$$m_{2,7} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (t_w \cdot 150 \text{ pcf}) \cdot \psi_{22}(x, z) \cdot \psi_{76}(x, z) dz dx$$

$$m_{3,4} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (t_w \cdot 150 \text{ pcf}) \cdot \psi_{11}(x, z) \cdot \psi_{21}(x, z) dz dx$$

$$m_{56} := m_{3,4}$$

$$m_{3,5} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (t_w \cdot 150 \text{ pcf}) \cdot \psi_{11}(x, z) \cdot \psi_{13}(x, z) dz dx$$

$$m_{3,6} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (t_w \cdot 150 \text{ pcf}) \cdot \psi_{11}(x, z) \cdot \psi_{23}(x, z) dz dx$$

$$m_{45} := m_{3,6}$$

$$m_{3,7} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (t_w \cdot 150 \text{ pcf}) \cdot \psi_{11}(x, z) \cdot \psi_{7}(x, z) dz dx$$

$$m_{4,6} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (t_w \cdot 150 \text{ pcf}) \cdot \psi_{21}(x, z) \cdot \psi_{23}(x, z) dz dx$$

$$m_{4,7} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (t_w \cdot 150 \text{ pcf}) \cdot \psi_{21}(x, z) \cdot \psi_7(x, z) dz dx$$

$$m_{5,7} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (t_w \cdot 150 \text{ pcf}) \cdot \psi_{13}(x, z) \cdot \psi_7(x, z) dz dx$$

$$m_{6,7} := \frac{1}{g} \cdot \int_0^L \int_0^{(h_1 + h_2)} (t_w \cdot 150 \text{ pcf}) \cdot \psi_{23}(x, z) \cdot \psi_7(x, z) dz dx$$

$$m_{i,j} := m_{j,i}$$

$$m = \begin{bmatrix} 0.044 & 0.025 & 5.996 \cdot 10^{-3} & 3.438 \cdot 10^{-3} & 5.996 \cdot 10^{-3} & 3.438 \cdot 10^{-3} & -0.011 \\ 0.025 & 0.157 & 3.438 \cdot 10^{-3} & 0.021 & 3.438 \cdot 10^{-3} & 0.021 & 0.021 \\ 5.996 \cdot 10^{-3} & 3.438 \cdot 10^{-3} & 0.012 & 7.037 \cdot 10^{-3} & -2.411 \cdot 10^{-3} & -1.382 \cdot 10^{-3} & -3.105 \cdot 10^{-3} \\ 3.438 \cdot 10^{-3} & 0.021 & 7.037 \cdot 10^{-3} & 0.043 & -1.382 \cdot 10^{-3} & -8.54 \cdot 10^{-3} & 5.788 \cdot 10^{-3} \\ 5.996 \cdot 10^{-3} & 3.438 \cdot 10^{-3} & -2.411 \cdot 10^{-3} & -1.382 \cdot 10^{-3} & 0.012 & 7.037 \cdot 10^{-3} & -3.105 \cdot 10^{-3} \\ 3.438 \cdot 10^{-3} & 0.021 & -1.382 \cdot 10^{-3} & -8.54 \cdot 10^{-3} & 7.037 \cdot 10^{-3} & 0.043 & 5.788 \cdot 10^{-3} \\ -0.011 & 0.021 & -3.105 \cdot 10^{-3} & 5.788 \cdot 10^{-3} & -3.105 \cdot 10^{-3} & 5.788 \cdot 10^{-3} & 0.083 \end{bmatrix} \text{ kips} \frac{\text{sec}^2}{\text{in}}$$