STABILITY FOR BRIDGE CABLE AND CABLE-DECK INTERACTION

Maria Rosaria Marsico, David J. Wagg and Simon A. Neild [1]

ABSTRACT

In this work the modal stability for a bridge-cable considering the relation between the second in plane and the first out-of-plane modal frequencies as $2\omega_1 = \omega_2$ will be developed. The analysis has been carried out on a cable 5.4 m long. We consider a simplified one degree of freedom cable-stay bridge model, representing a bridge deck movement, which interacts with a particular cable in the structure. The first out-of-plane and the second in-plane mode are known to have modal cross-coupling. Analytical stability boundaries can be derived to identify the point at which modal coupling destabilizes the out-of-plane mode. The experimental results have been analyzed using a model of the cable in the Earthquake Engineering Laboratory at the University of Bristol. In particular the cable has been tested using real–time dynamic substructuring which is a form of hybrid testing. Analytical and experimental results were compared and a good correlation was obtained.

INTRODUCTION

Cable-stay bridges are very common in the world especially because they are a wonderful expression of the antique and modern architecture and very useful too. These bridges have three fundamental structural elements: pylons, deck and cables and their interaction is very complex especially because the large number of the cables connected with the deck. Also the non-linearity of the cable behaviour and the vibrations transferred from the deck to the cables create a complicated system. Is very difficult to reproduce the bridge behaviour in the Laboratory considering all the cable so firstly we decide to take in account just one cable moved by deck's vibrations. Due to interactions between the deck and cable the dynamics are complex so we simulate them using a new technique called Real-Time Dynamic Substructuring (RTDS). The principle is very similar to a hybrid-test. A part of the specimen is physically tested (the cable) and a part is modelled numerically (the deck) (Marsico *et al.* 2009). A Matlab-Simulink model is used to simulate the deck behaviour and to give us the possibility to change its characteristics just using an input by a computer. This technique lets us study a cable and connect it with many different substructures.

In this work we analyze only one cable's mode shapes under deck excitations. The scope is defining the instability point that is considered when the cable response is out-of-plane.
The scope is to identify the excitation amplitudes at which out-of-plane motion is triggered as a function of forcing frequency. The point at which the out-of-plane motion is triggered there is a loss of local stability of the zero-amplitude response, we term this the instability point. For this research we are interested on three fundamental modes in order to define the relation between in-plane and out-plane modes. In particular the most significant parametric excitation occurs at the 2:1 resonance ratio, that means the second out-of-plane is very close the first in-plane frequency. This phenomenon was observed analytically and experimentally. The tests have been conducted

[1] *Department of Mechanical Engineering, University of Bristol, Bristol BS9 3NW, UK*

at the Earthquake Engineering Laboratory at University of Bristol on a cable long 5.4 m where the deck was simulated numerically.
A comparison of the results shows a good agreement between the analytical and the experimental model.

THE THEORETICAL MODEL

The model of the cable-deck interaction system is shown schematically in Figure 1, where u ,v and w are axial, out-of-plane transverse and in-plane transverse displacements of the cable respectively and θ is the angle of inclination measured from the horizontal line in the gravity plane. The cable is fixed at the upper end and an oscillator is included at the lower end. This oscillator represents a particular global mode of vibration of the bridge that is particularly likely to interact with the cable since its frequency is close to twice the first out-of-plane natural frequency of the cable. For the cable dynamics we adopt the model equations derived by Warnitchai *et al.* (1995) which includes the effects of modal coupling and of support motion at both ends of the cable.

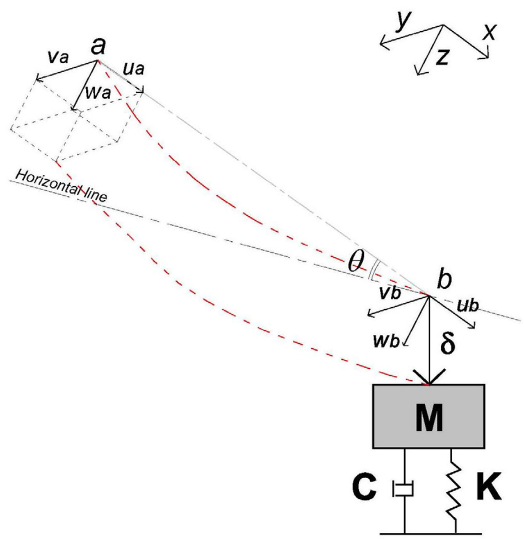

Fig. 1. Cable-deck interaction phenomena: cable-sdof system.

The following set of equations capture the modal behaviour of a taut cable subject to support movement. Firstly the linearized cable dynamics were considered, under the assumptions that sag is small compared to the length of the cable, the dynamics along the cable are insignificant, and the amplitude of vibration is small compared with the sag such that the compatibility equation can be linearized. The mode shapes for the linearized system in both the in-plane and out-of-plane directions were calculated. These modes shapes were then used in the derivation of the modal equations of motion where the assumption that the amplitude of vibration is small compared to the sag is relaxed by using a nonlinear compatibility expression. or the out-of-plane and anti-symmetric in-plane modes, the mode shapes for the linearized system are assumed to be sinusoidal. The symmetric in-plane modes are more complex (Irvine 1981) calculating the nonlinear terms the symmetric in-plane modes are also taken to be sinusoidal, a reasonable assumption for systems with taut cables, given that the nonlinear terms are expected to be small compared to the linear terms. The resulting modal representation of the out-of-plane cable motion may be expressed as

$$
\begin{aligned}
& m_{yn}\left(\ddot{y}_n + 2\xi_{yn}\omega_{yn}\dot{y}_n + \omega_{yn}^2 y_n\right) + \sum_k v_{nk} y_n \left(y_k^2 + z_k^2\right) + \\
& \sum_k 2\beta_{nk} y_n z_k + 2\eta_n (u_b - u_a) y_n + \zeta_n \left(\ddot{v}_a + (-1)^{n+1} v_b\right) = F_{yn}
\end{aligned} \tag{1}
$$

and the in-plane cable motion as

$$
\begin{aligned}
& m_{zn}\left(\ddot{z}_n + 2\xi_{zn}\omega_{zn} z_n + \omega_{zn}^2 z_n\right) + \sum_k v_{nk} z_n \left(y_k^2 + z_k^2\right) + \sum_k 2\beta_{nk} z_n z_k + \\
& \sum_k \beta_{kn}\left(y_k^2 + z_k^2\right) + 2\eta_n (u_b - u_a) z_n + \zeta_n \left(\ddot{w}_a + (-1)^{n+1} \ddot{w}_b\right) - \alpha_n \left(\ddot{u}_b - \ddot{u}_a\right) = F_{zn}
\end{aligned} \tag{2}
$$

where y_n and z_n are the out-of-plane and in-plane generalized displacement of the cable in the n^{th} mode respectively; subscripts a and b denote the top and bottom anchorage points respectively; $\mathrm{m}_{yn}=\mathrm{m}_{zn}=\mathrm{m}$ is the modal mass ($\mathrm{m}=\rho\, A\, L/2$); L is the cable length; σ_s is the cable static stress; λ^2 is Irvine's parameter (Irvine 1981), A is the cross section area, ρ is the density and E is the Young's Modulus. The effective axial modulus of the cable, E_q, the distributed weight perpendicular to the cable cord, γ, and the parameters k_n, ν_{nk}, β_{nk}, η_n, α_n, λ, and F_{yn} and F_{zn} which represent external cable loading in the y and z direction respectively are given in Marsico *et al.*(2009). The out-of-plane and in-plane natural frequencies, ω_{yn} and ω_{zn} respectively, are given by

$$
\omega_{yn} = \frac{n\pi}{L}\sqrt{\frac{\sigma_s}{\rho}} \quad \text{and} \quad \omega_{zn} = \frac{n\pi}{L}\sqrt{\frac{\sigma_s}{\rho}\left(1 + k_n\right)} \tag{3}
$$

The factor k_n represents the effect of sag and for even n, k_n is zero. For n odd, k_n is a positive value at most of the order of $1\text{x}10^{-2}$. Therefore as a first approximation we can neglect k_n and calculate the in-plane natural frequencies with the same formula used for out-of-plane frequencies. It is assumed that damping can be modelled as viscous with modal damping ratios ξ_{zn} and ξ_{yn} obtained experimentally. The deck is modelled as a single degree-of-freedom system, so the resulting equation of motion is:

$$
M\ddot{\delta} + C\dot{\delta} + K\delta + T\sin(\theta) = F_e \tag{4}
$$

where M, C and K are mass, damping and stiffness of the oscillator (i.e., deck) and δ is the deck displacement. T is the dynamic cable tension, which is obtained from the cable dynamic stress. We apply $F_e=F\sin(\Omega t)$ where F is the amplitude of the excitation force and Ω is the forcing frequency for the cable excited vertically at the bottom, anchorage (point b).

Finally the dynamic tension (there is also a static component due to pre-tensioning and self weight) in the cable may be calculated from the compatibility equation (Lorenzo et al. 2003)

$$
T = EA\varepsilon = EA\left[\frac{E_q}{E}\left(\frac{u_b - u_a}{L}\right) + \frac{\gamma}{\pi\sigma_s}\sum z_n \frac{1}{n}\left(1 + (-1)^{n+1}\right) + \frac{\pi^2}{4L^2}\sum n^2\left(y_n^2 + z_n^2\right)\right] \tag{5}
$$

The oscillator is excited vertically by a external force of amplitude F and frequency Ω. Noting that $\omega_{y1} \neq \omega_{z1}$ and $\omega_{z2} = \omega_{y2} = 2\omega_{y1}$, due to cable sag, we write $\omega_1 = \omega_{y1}$ and $\omega_2 = \omega_{y2} = \omega_{z2}$. Assuming negligible response of other modes in the studied frequency range Ω close to ω_1, we can then write the equation of motion for the first in-plane mode. Finally the deck equation, along with the compatibility equation, also can be written (see Marsico *et al.* 2009 for details). From these equations we can note that the deck frequency, ω_g is effected by both the deck stiffness and the cable stiffness.

Using the analysis proposed in Gonzalez-Buelga e*t al.* (2007) but extending it to include the deck contribution, we scale the equations using the small parameter ε, such that they are in the standard Langrage form. The forcing frequency Ω and the oscillator (i.e. global mode) frequency ω_g are close to twice the first out-of-plane natural frequency, therefore we write $\Omega = \omega_2(1+\mu)$. Using this, taking into account that $\omega_2 = 2\omega_1$ and applying the time transform $\tau = (1+\varepsilon\,\mu)t$, we can write the scaled equations of motions. These equations are now in a form which can be averaged. See Verhulst *et al.* (1996) for more details on the averaging method.

Localized Stability

In this section we examine the Equations to assess the stability boundary of the semi-trivial solution which physically corresponds to the point at which the cable starts to have out-of-plane response when both the input and previous response were in-plane (Marsico *et al.* 2009). The external excitation will lead directly to in-plane cable and oscillator motions. With increasing excitation amplitude either of the out-of-plane modes can be excited, marking the boundary of the semi-trivial solution in the chosen parameter space. For excitation of either out-of-plane modes there must be localized instability about the zero amplitude response for that mode. To find the boundary of the semi-trivial solution in parameter space we therefore examine the localized stability of each out-of-plane mode about the zero point assuming that the other out-of-plane mode has zero averaged amplitude. For the first out-of-plane mode we can write

$$\begin{Bmatrix} y_{1ca}' \\ y_{1sa}' \end{Bmatrix} = \varepsilon \begin{bmatrix} ab \\ cd \end{bmatrix} \begin{Bmatrix} y_{1ca} \\ y_{1sa} \end{Bmatrix} \tag{6}$$

where

$$a = -\xi_{y1}\omega_1 + \frac{N_1\delta_{sa}}{4\omega_1} \qquad b = -\frac{N_1\delta_{ca}}{4\omega_1} - \mu\omega_1 + \frac{W_{12}Z_{2a}^2}{4\omega_1}$$

$$c = -\frac{N_1\delta_{ca}}{4\omega_1} + \mu\omega_1 + \frac{W_{12}Z_{2a}^2}{4\omega_1} \qquad d = -\xi_{y1}\omega_1 - \frac{N_1\delta_{sa}}{4\omega_1}$$

where we have set the y_2 mode amplitude to zero and neglected the higher order y_{1ca} and y_{1sa} terms as we are considering the stability about the y_{1a}=0 point (Gonzalez-Buelga *et al.* 2008). The resulting eigenvalues of the matrix in Eqn.(6), denoted χ (where we apply the scaling $\chi \rightarrow \varepsilon\chi$, are given by the roots of

$$16\omega_1^2\chi^2 + 32\xi_{y1}\omega_1^3\chi + W_{12}^2 Z_{2a}^4 - 8W_{12}\mu\omega_1^2 Z_{2a}^2 + 16\omega_1^4\left(\mu^2 + \xi_{y1}^2\right) - N_1^2\Delta^2 = 0$$

We note that initially when the excitation amplitude is small (such that Δ and Z_{2a}^2 are small) the eigenvalues of the matrix have negative real parts and hence the stable solution set is from zero excitation up to the boundary at which the real part of one of the eigenvalues is zero. This stability boundary is given by

$$W_{12}^2 Z_{2a}^4 - 8W_{12}\mu\omega_1^2 Z_{2a}^2 + 16\omega_1^4\left(\mu^2 + \xi_{y1}^2\right) - N_1^2\Delta^2 = 0 \qquad \text{`(8)}$$

where $\Delta^2 = \delta_{ca}^2 + \delta_{sa}^2$. Using the same technique for the second out-of-plane mode and noting that ω_2=2ω_1 and W_{22}=4W_{12}. The stability boundary is defined by:

$$3W_{12}^2 Z_{2a}^4 - 32W_{12}\mu\omega_1^2 Z_{2a}^2 + 64\omega_1^4\left(\mu^2 + \xi_{y2}^2\right) = 0 \qquad (9)$$

Note that for $\mu < \sqrt{3\xi_{y2}}$ the second out-of-plane mode is stable about the zero amplitude position for all Z_{2a} and hence for all oscillator amplitudes Δ. The steady state displacement of the oscillator is:

$$\delta_{sa} = \frac{-\xi_g\omega_g\omega_2\hat{F}}{2\left(\left(\xi_g\omega_g\omega_2\right)^2 + \left(\mu\omega_2^2 - \frac{\sigma}{2}\right)^2\right)} \qquad \delta_{ca} = \frac{\left(\mu\omega_2^2 - \frac{\sigma}{2}\right)^2\hat{F}}{2\left(\left(\xi_g\omega_g\omega_2\right)^2 + \left(\mu\omega_2^2 - \frac{\sigma}{2}\right)^2\right)} \qquad (10)$$

and the response in the in-plane second mode is:

$$16\omega_1^4 B^2\Delta_a^2 = 64\omega_1^4\left(\mu^2 + \xi_{z2}^2\right)Z_{2a}^2 - 48\omega_1^2\mu W_{12} Z_{2a}^4 + 9W_{12}^2 Z_{2a}^6 \qquad (11)$$

In order to calculate the first out-of-plane stability boundary, Eqns. (7) and (11) have to be solved simultaneously. The static tension of the cable is 286 N. For low amplitude deck vibration, 0.5% damping is typical, but there is evidence that the damping is amplitude dependent, so as we are assuming nonlinear interactions, due to larger amplitude excitation, a value of 1% is assumed. The natural frequencies of the cable are detailed in Tab.1. From experimental data the modal damping over the range of oscillation amplitude of interest was estimated to be ξ=0.2% for all modes.

Table 1. Cable natural frequencies.

	ω_{y1} [rad/sec]	ω_{y2} [rad/sec]	ω_{z1} [rad/sec]	ω_{z2} [rad/sec]
Experimental	3.25·2π	6.51·2π	3.34·2π	6.51·2π
Theoretical	3.25·2π	6.51·2π	3.32·2π	6.51·2π

THE EXPERIMENTAL MODEL

Many tests at the Earthquake Engineering Laboratory of Bristol University have been carried out on the cable equipped in the Lab (Figure 2). It is 5.4 m long with diameter 0.0008 mm and to increase its weight 22 lead masses have been applied, spaced each 250 mm, except the first one at the bottom which distances 200 mm. These masses will be very useful to acquire data during the tests because they give us the possibility to visualise the discrete points for the mode vibrations. It is fixed on the top where manually is possible to increase or decrease the static tension until the value we are interested. The tension is measured in volt by a load cell and visualized in the acquisition box. This value can be checked using software for control installed on the computer for the input data giving back the converted value in Newton. At the top a single axial S shape load cell measures the tension in the cable. At the bottom it is connected with a steel beam representing the behaviour of the deck in a common cable-stay bridge through a multiaxial (six DOF) load cell and a vertical LVDT with limit displacement of ±10mm. The actuator has 10 kN maximum force and ±150 mm displacement. Therefore it can also measure the displacement thanks to the internal LVDT but because the reduced dimension of the cable is better using the smaller one connected with the deck.

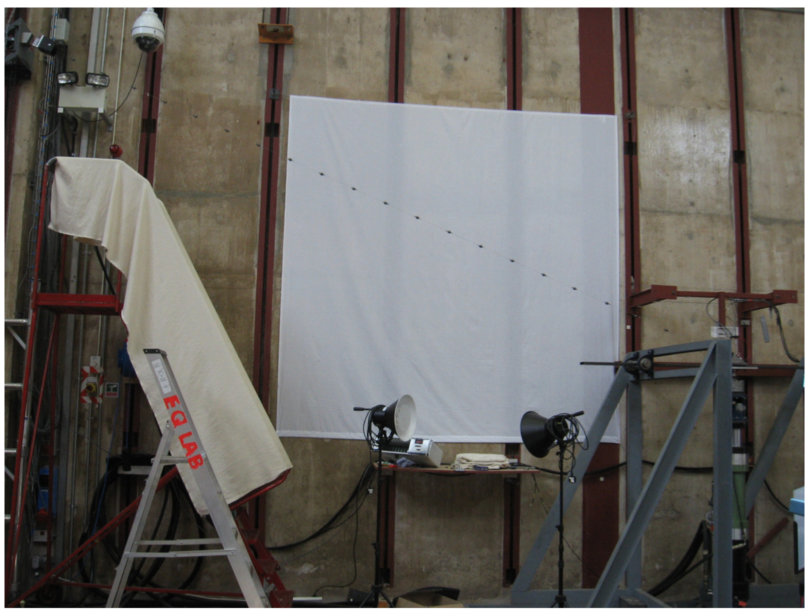

Fig. 2. Experimental test set up.

A very complex system is used for the input and output data (Figure 3). A Simulink file was created to connect the given input data, and record the results. A big influence is given by the delay from the input and acquisition systems. The input data are inserted in the computer installed in the Laboratory and in particular we can set the amplitude, frequency and delay for different kind of tests. This computer is connected with the set up for the test model and the acquisition instruments so we can verify the values applied. Using a monitoring system the data go to another pc able to measure the difference and optimize the results. All the data are acquired in another pc using the Video Gauge System (Figure 4). In particular two cameras installed in the Laboratory are used. The first one is frontal and is able to record and visualize the in plane motion and the second one is lateral and gives us the out-of-plane motion behaviour.

So the Video Gauge software acquires the masses' movements and we are able to define the vibration modes of the cable. Another software installed on the same PC renders the relation between frequency and amplitude considering the displacement in pixels because it is a video acquisition systems. A program was developed to convert the data from pixel to the SI units and it's very important to display the displacement.

The actuator imitates the deck and excites the cable with a sine wave input. It works using the oil pump with 100 l/min capacity, pressure 23 MPa and controlled by the command cabin in the Laboratory. From this cabin we can set the oil pressure for the tests on the cable.

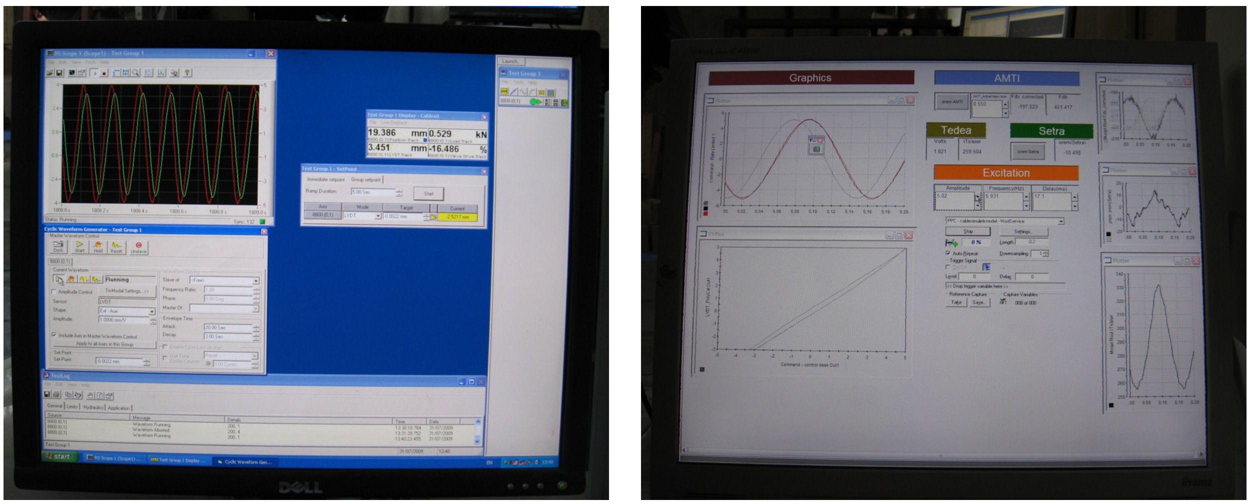

Fig. 3. Input system.

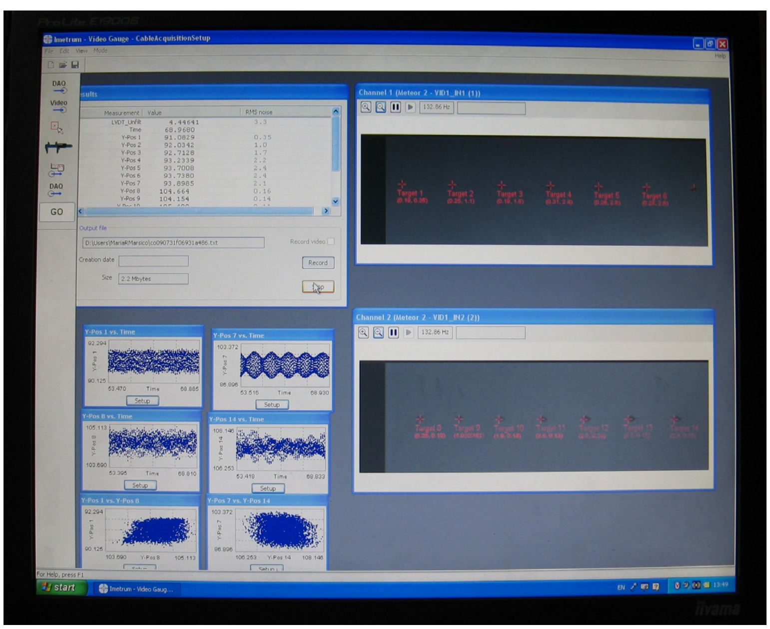

Fig. 4. Video gauge system.

AGREEMENT BETWEEN THEORETICAL AND ANALYTICAL MODEL

The analytical stability boundaries picture is plotted in Figure 5 where the solid red line represents the analytical results for different value of μ (where μ is q-1 and q is the ratio between ω_g -oscillator frequency- and ω_2 -second out of plane frequency-), the green dots represent stable and the blue diamantes unstable experimental points. It is very interesting to observe that the instability for Ω/ω_1 >2.00 are experimentally for both verified the second out of plane modes and first in plane modes. We can note that close to the resonance point the stability region becomes very small. Also we note that in the region where the turning point is below the theoretical stability boundaries, see Figure 5, the Δ required for instability of the semi-trivial solution is governed by the theoretical stability boundary rather than the turning point relationship and hence the theoretical stability boundaries are conservative.

Currently the most widely used stability curve when studying 2:1 resonance is the one presented by Lilian and Pinto da Costa (1994) which is used in practical bridge design recommendations, such as those produced by SETRA (2002), to provide guidance as to whether the expected cable anchorage motions would be large enough to initiate parametrically excited vibrations. The equations of the stability boundary in both works are found from a linear one-degree-of-freedom Mathieu-Hill type equation. Since they reduce the study to a single degree of freedom they calculate y_1 and y_2 boundaries separately, the first excited in 2:1 resonance, the second in 1:1 resonance (due to external support excitation rather than internal resonance with the second in-plane mode). For the y_1 mode excited close to 2:1 resonance, (Lilien et al. 1994) states that the stability boundary is given by

$$\widehat{\Delta} = 2\frac{\varepsilon_s}{\sin(\theta)}\sqrt{\left[\left(\frac{\Omega}{2\omega_1}\right)^2 - 1\right]^2 + 4\xi^2\left[\frac{\Omega}{2\omega_1}\right]^2} \tag{12}$$

An equivalent expression is given by SETRA (2002) which is virtually the same for $\Omega/(2\omega_1)\approx 1$. The equations, which have been verified experimentally in the section above, predict that the minimum excitation amplitude occurs away from resonance although the minimum amplitude remains approximately the same. In addition, the excitation amplitude to induce a response in the y_1 mode is significantly less than that predicted by Lilien (1994) and SETRA (2002) for frequencies above resonance. This shifting of the minimum and the reduction in amplitude of the higher frequency sides of the stability boundary are a direct consequence of the hardening that cables experience due to the geometric cubic nonlinearity from concurrent vibrations in the second in-plane mode. This nonlinearity is not taken into account in previous stability models, and as a result the match with experimental data (as shown in the previous section) will be reduced. The reduction in amplitude of the curve shows clearly that parametric resonance can occur for much smaller anchorage motions that previously predicted when Ω/ω_2>1.

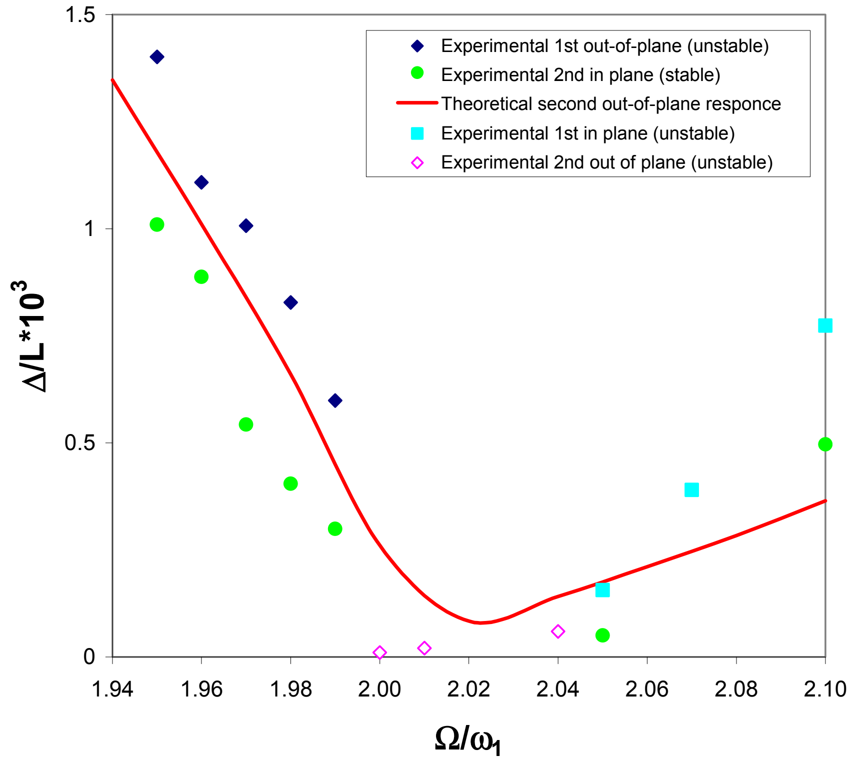

Fig. 5. Stability boundaries: analytical and experimental results.

CONCLUSIONS

In this paper we have presented the interaction between in-plane and out-of-plane modes for a bridge cable. It can move in different modes (in-plane and out-of-plane) but we have concentrated just on three modes (second-in-plane, first-out-of-plane and second out-of-plane) because their very strictly interactions in terms of frequencies. The fixed cable supported at the upper end and connected at the lower end to the deck (as a single degree of freedom system) has been modelled and the equations to describe the modes have been defined.

By including the model coupling terms, and using averaging, the three mode model has been used to explain some subtle dynamic behaviour which occurs around the 2:1 internal resonance of the in and out-of-plane modes.

A part of this work consists in tests performed using a cable 5.4m long with attached masses at fixed intervals to simulate approximately its real weight. The Real-Time Dynamic Substructuring (RTDS) method has been adopted to carry out the tests and in particular the vertical excitation force has been applied by an actuator to give input at the lower support.

The series of tests was conducted to observe the onset of oscillations in the out-of-plane modes, and these were compared with analysis and simulation from the three mode model.

It can be noted the theoretical line represents very well the experimental results: it means that the procedure used in this work can be considered as a method of extending traditional Laboratory

test techniques. This also demonstrates the importance of including the nonlinear coupling terms when studying the stability boundaries close to the 2:1 resonance region.
Future work may include using devices close the bottom of the cable or may be using more cables and study their interaction.

ACKNOWLEDGMENTS

The authors would like to acknowledge the support of the EPSRC: Dr. M.R. Marsico is supported by grant EP/F030711/1. We also thank Dr M.S. Dietz for sharing his experimental expertise when setting up the tests.

REFERENCES

Gonzalez-Buelga, A., Wagg, D.J., and Neild, S.A. 2007. Parametric variation of a coupled pendulum-oscillator system using real time dynamic substructuring, *Structural Control and Health Monitoring*, **14**(7):991–1012.

Gonzalez-Buelga, A., Neild, S.A., Wagg, D.J., and Macdonald, J.H.G. 2008. Modal stability of cables subjected to dynamic loading, *Journal of Sound and Vibration*, **318**(3):565–579.

Irvine H.M. 1981. *Cable structures*, MIT Press.

Lilien, J.L., and Pinto da Costa A. 1994. Vibration amplitudes caused by parametric excitation of cable-stayed structures, *Journal of Sound and Vibration*, **174**(1): 69–90.

Lorenzo, R., and Macdonald, J. 2003. Experimental validation of a simplified cable-stayed bridge model exhibiting autoparametric resonance, *Proceeding, 5th Int. Symposium Cable Dynamics*, S. Margherita Ligure, Italy.

Marsico, M.R., Neild, S.A., Gonzalez-Buelga, A., Wagg, D.J. 2009. Interaction between in-plane and out- of-plane cable mode for a cable-deck system, *Proceedings, ASME 2009 Design Engineering technical Conference & Computers and Information in Engineering Conference (DETC2009)*, San Diego, California.

SETRA 2002. Cable Stays: Recommendations of French interministerial commission on Prestressing, *Center des Techniques des Oeuvres d'Art, Bagneux Cedex*, France, 93–96.

Verhulst, F. 1996. *Nonlinear Differential Equations and Dynamical Systems*, Springer.

Warnitchai, Y., Fujino, T., and Susumpov, A. 1995. A nonlinear dynamic model for cables and its application to a cable structure-system, *Journal of Sound and Vibration*, **187**(4):695–712.