

# IMPLICIT NUMERICAL INTEGRATION FOR HYBRID SIMULATION OF COMPLEX STRUCTURAL SYSTEMS

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## ABSTRACT

A fully implicit iterative integration procedure is developed for local and geographically distributed hybrid simulation of the seismic response of complex structural systems with distributed nonlinear behavior. The purpose of this procedure is to allow experimental elements in simulations using existing fully implicit integration algorithms designed for pure numerical simulations. The implementation difficulties of implicit integration are addressed at the element level by introducing a safe iteration strategy and using an efficient procedure for online estimation of the experimental tangent stiffness matrix. In order to avoid physical application of iterative displacements, the required experimental restoring force at each iteration step is estimated from polynomial curve fitting of recent experimental measurements. The tangent stiffness matrix is estimated using readily available experimental measurements and by a classical diagonalization approach that reduces the number of unknowns in the matrix. Numerical and hybrid simulations are used to demonstrate that the proposed procedure provides an efficient method for implementation of fully implicit numerical integration in hybrid simulations of complex nonlinear structures. The hybrid simulations presented include distributed nonlinear behavior in both the numerical and experimental substructures.

## INTRODUCTION

Hybrid simulation combines numerical and experimental methods for cost-effective, large-scale laboratory testing of structures under simulated earthquake loading (Mahin et al. 1989; Shing et al. 1996; Takanashi and Nakashima 1987). The equation of motion is expressed for the combined experimental and numerical components and solved using a time-stepping integration procedure as in numerical simulations. Explicit integrators, such as the Central Difference Method, are simple to implement in a hybrid test, but their conditional stability limits their application to simple structural models. A combination of non-iterative implicit and explicit integration algorithms (Dermitzakis and Mahin 1985), including the operator-splitting method (Nakashima et al. 1990), offer improved stability and accuracy, but use the initial stiffness matrix to predict the nonlinear response of the specimen. A tangent stiffness matrix has been proposed to improve this correction (Ahmadizadeh and Mosqueda 2008b).

Fully implicit integration algorithms are widely used in pure numerical simulations of the seismic response of structures for their superior stability and accuracy at larger time steps compared to explicit methods. However, the direct application of implicit integration algorithms to hybrid simulation has been partially limited by the requirement to iterate with experimental substructures and difficulties in estimating the experimental tangent stiffness matrix. Past implementations have relied on conservative iterations that ensure each iteration results in a

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monotonically increasing path to the converged displacement (Shing et al. 1991), imposing iterations through a feedback loop between the integrator and the experimental substructure (Shing et al. 2006; Wu et al. 2007), or treating the interface forces between the experimental and numerical substructures as constant external forces (Ghaboussi et al. 2006; Schneider and Roeder 1994). More recently, Pan *et al.* (2005) conducted a distributed test using finite element software and fully implicit integration algorithms that capture the instantaneous behavior of a single-degree-of-freedom (SDF) experimental substructure.

In order to extend the capabilities of hybrid simulation to complex structural systems with nonlinear behavior distributed throughout the structural model, a fully implicit integration method is presented that is compatible with experimental substructures. The implementation issues of dealing with the experimental substructure are handled by introducing a safe iteration strategy and a procedure for estimation of the experimental tangent stiffness matrix. The effectiveness of the proposed integration method is demonstrated through actual hybrid simulations that show the ability of this approach to accurately capture the behavior of the experimental substructure. The accuracy of the simulation is measured by computing the overall energy balance of the simulation and the energy errors introduced by experimental and numerical errors (Ahmadizadeh and Mosqueda 2008a).

## NUMERICAL INTEGRATION

In a hybrid simulation, the equation of motion of the combined numerical and experimental structural model can be expressed as:

$$\mathbf{M}\mathbf{a} + \mathbf{C}\mathbf{v} + \mathbf{r} + \mathbf{r}^e = \mathbf{f} \quad (1)$$

in which  $\mathbf{M}$  and  $\mathbf{C}$  are mass and damping stiffness matrices of the numerical substructure,  $\mathbf{f}$  is the external force vector,  $\mathbf{v}$  and  $\mathbf{a}$  are velocity and acceleration vectors, respectively,  $\mathbf{r}$  is the restoring force from the numerical substructure, and  $\mathbf{r}^e$  is the restoring force measured in the experimental substructures. Here, it is assumed that the nonlinear restoring forces of the numerical and experimental substructures are essentially strain-dependent.

Time-stepping integration procedures solve the equilibrium equation of motion (1) at discrete time steps combined with the kinematic relations between the states. For example, the finite difference kinematic relations in the Newmark's Beta integration procedure (Newmark 1959) are:

$$\mathbf{d}_n = \mathbf{d}_{n-1} + \Delta t \mathbf{v}_n + \left(\frac{1}{2} - \beta\right)(\Delta t)^2 \mathbf{a}_{n-1} + \beta(\Delta t)^2 \mathbf{a}_n \quad (2)$$

$$\mathbf{v}_n = \mathbf{v}_{n-1} + (1 - \gamma)\Delta t \mathbf{a}_{n-1} + \gamma\Delta t \mathbf{a}_n \quad (3)$$

where  $\mathbf{d}$  is the displacement vector,  $\Delta t$  is the integration time step,  $n$  is the integration step number, and  $\beta$  and  $\gamma$  are integration parameters that affect the stability and accuracy of the integration scheme, such as numerical damping and period elongation. Numerical damping in the integration algorithms is sometimes used to suppress the response of high-frequency modes of vibration that may be excited by experimental errors and measurement noise. In order to provide numerical damping without affecting the accuracy of the integration algorithm, Hilber *et al.* (Hilber et al. 1977) proposed the  $\alpha$ -method for numerical integration. This algorithm is used here with the restoring force:

$$\mathbf{r}_n = \mathbf{r}_{n-1} + \mathbf{K}_n (\mathbf{d}_n - \mathbf{d}_{n-1}) \quad (4)$$

In hybrid simulations, implementation of a fully implicit iterative integration procedure is difficult because of: (i) the required online estimation of experimental tangent stiffness matrix, (ii) the potential for unwanted damage to the experimental substructure through application of iterative displacements, and (iii) uncertainties in convergence of iterative scheme in nonlinear simulations involving numerical and experimental errors.

Iterative integration procedures are preferred for numerical simulation of large systems with distributed nonlinearities. Here, it is attempted to provide an efficient implicit integration procedure for hybrid simulations of large structural systems that can accurately capture large nonlinearities distributed throughout the structure.

### Proposed Implicit Integration Algorithm

Each step of the proposed implicit integration procedure begins with calculation of the desired displacement vector. For this purpose, the predictor displacement vector  $\tilde{\mathbf{d}}_n^l$  is obtained from Equation (2) by temporarily setting  $\beta = 0$ :

$$\tilde{\mathbf{d}}_n = \mathbf{d}_{n-1} + \Delta t \mathbf{v}_n + \frac{1}{2}(\Delta t)^2 \mathbf{a}_{n-1} \quad (5)$$

and transforming the result to actuator local coordinate system. This displacement vector is applied on the experimental substructure and measurements are fed back to the numerical simulation module. Next, these measurements are used to solve the time discrete form of (1) iteratively. The restoring force vector is updated in each iteration according to the new iterative displacement vector and then used to estimate the stiffness matrices of numerical and experimental substructures using:

$$\mathbf{r}_n^{e,l} = \tilde{\mathbf{r}}_n^{e,l} + \mathbf{K}_n^{e,l} (\mathbf{d}_n^l - \tilde{\mathbf{d}}_n^{l,m}) \quad (6)$$

In (6),  $\tilde{\mathbf{r}}_n^{e,l}$  is the measured experimental restoring force,  $\tilde{\mathbf{d}}_n^{l,m}$  is the measured displacement vector, and  $\mathbf{K}_n^{e,l}$  is the experimental tangent stiffness matrix, all expressed in actuator local coordinates. Note that in the conventional operator-splitting method, this correction is made once in each integration step using the initial stiffness matrix of the experimental substructure. Here, iterations are repeated until a specified convergence criterion is satisfied. The implementation procedures of the proposed algorithm are explained further in the following sections.

### Estimation of Experimental Restoring Forces

The experimental restoring forces can be updated without physically imposing iterative displacements by interpolation and extrapolation of the most recent force and displacement measurements (Ahmadizadeh et al. 2007). The experimental restoring forces are estimated using second-order polynomials fitted to the last four points of force and displacement measurements of each actuator. As shown in Fig. 1, the iterative displacement in actuator coordinate system is first used to estimate its corresponding time at which the displacement was achieved. This time value is then substituted in the force polynomial to give an estimate of the restoring force. The increments of displacements and estimated restoring forces from previous step are used to update the experimental tangent stiffness matrix following the procedure outlined in the next section.

It should be noted that the accuracy of the estimated restoring forces is reduced as the extrapolation time increases. For this reason, limits should be placed on the time parameter to avoid excessive extrapolations. It is proposed to keep the experimental tangent stiffness matrix unchanged when force estimation is not possible within the allowed time variation range.

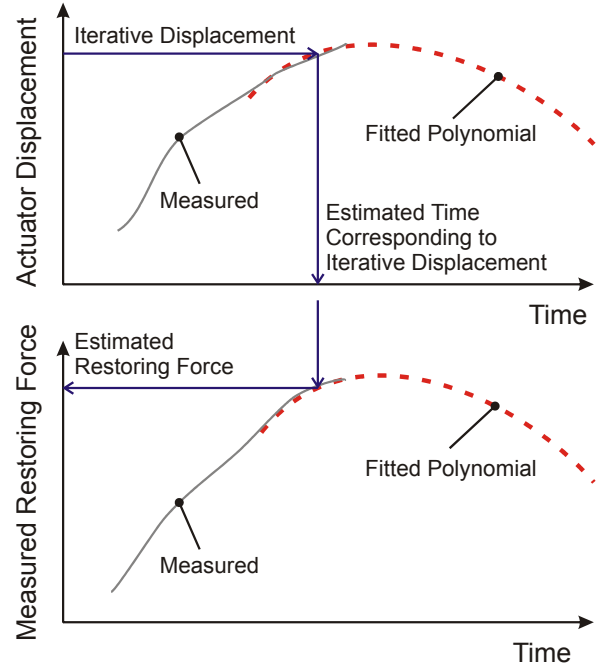
The above-mentioned approach eliminates the need for physical application of iterative displacements on experimental substructure. As a result, potential displacement reversals during iterations will not damage the experimental substructure or lead to erroneous experimental energy dissipation. Importantly for geographically distributed testing, data exchange between numerical and experimental subsystems occurs only once in each integration step.

#### Online Estimation of Tangent Stiffness Matrix

Methods for updating the tangent stiffness matrix of the numerical substructure during the simulation are well established (McGuire et al. 2000). These methods normally take advantage of information about element internal forces and plasticity states, which are not available for experimental substructures. Here, an efficient method for estimation of experimental tangent stiffness matrix is presented. It should be considered that in a typical hybrid simulation, a large portion of the test structure is numerically modeled, while the experimental substructures consist of only a few degrees of freedom.

The online estimation of experimental tangent stiffness is difficult due to the contamination of measurements required for the computation, and the limited data available for establishing the full stiffness matrix. To address the first issue, the experimental tangent stiffness matrix is updated only in steps with displacement increment sufficiently larger than a specified threshold selected as the greater of: 10 times the RMS of displacement noise or a value that results in a force (using initial stiffness) 10 times greater than the RMS of force noise (Ahmadizadeh and Mosqueda 2008b). The stiffness matrix remains unchanged in integration steps with small displacement increments.

A second issue in online estimation of experimental tangent stiffness matrix is the limited number of measurements (equal to the number of actuators and load cells,  $m$ ) compared to the number of elements of the stiffness matrix (up to  $m^2$ ). In order to address this problem, Ahmadizadeh and Mosqueda (2008b) proposed reducing the number of unknowns using the information regarding the element configuration and geometric properties of the experimental substructure. This was achieved using a transformation matrix  $\mathbf{T}_p$  that reduced the  $m \times m$  stiffness



**Fig. 1. Estimation of measured force at target iterative displacement using polynomials fitted to experimental data**

matrix of the experimental substructure to a diagonal matrix  $\mathbf{P}_n$  in an intrinsic coordinate system, consisting of parameters that constitute the sources of resistance and nonlinear behavior of the experimental substructure:

$$\mathbf{K}_n^{e,l} = \mathbf{T}_p^T \mathbf{P}_n \mathbf{T}_p \quad (7)$$

where  $\mathbf{K}_n^{e,l}$  is the experimental tangent stiffness matrix at step  $n$  expressed in actuator local coordinate system. By dividing the incremental forces by incremental displacements in parameter coordinate system and using the above transformation, the tangent stiffness can be obtained in actuator local coordinates. The transformation  $\mathbf{K}_n^e = \mathbf{T}^T \mathbf{K}_n^{e,l} \mathbf{T}$  can then be used find this stiffness matrix in global coordinates, where  $\mathbf{T}$  transforms displacements from global to actuator local coordinate system.

The above procedure is generalized here by following the classical method of diagonalization of stiffness matrix. An  $m \times m$  matrix is diagonalizable if it has  $m$  linearly-independent eigenvectors. This is the case when the matrix has  $m$  distinct eigenvalues (Greenberg 1998). Given these conditions, the following relation can be used to diagonalize the tangent stiffness matrix at step  $n$ :

$$\mathbf{P}_n = \Phi_n^{-1} \mathbf{K}_n^{e,l} \Phi_n \quad (8)$$

in which:

$$\Phi_n = [\phi_1 \phi_2 \cdots \phi_m]_n \quad (9)$$

is a matrix of normalized eigenvectors (modal matrix) of the local stiffness matrix at step  $n$ ,  $\mathbf{K}_n^{e,l}$ . In addition to the above, the symmetry of the stiffness matrix results in the orthogonality of eigenvectors (or the dynamic mode shapes with an identity mass matrix) (Chopra 2001), which further facilitates the diagonalization process by changing Equation (8) to:

$$\mathbf{P}_n = \Phi_n^T \mathbf{K}_n^{e,l} \Phi_n \quad (10)$$

Hence, a general choice of the transformation matrix  $\mathbf{T}_p$  is the transpose of the eigenvectors matrix:

$$\mathbf{T}_p = \Phi_n^T \quad (11)$$

Since this transformation matrix is square, the number of stiffness parameters to be estimated in the diagonal matrix will be equal to  $m$ .

It should be noted that since the stiffness matrix is updated in each integration step, the mode shapes may also change, and the transformation matrix needs to be updated accordingly. That is, an eigenvalue problem involving the tangent stiffness matrix should be solved in each integration step. Furthermore, since the stiffness matrix changes within an integration step after being used to derive the transformation matrix, an iterative procedure is necessary to simultaneously update the stiffness and transformation matrices. In order to avoid these iterations, the tangent stiffness matrix in the previous step  $\mathbf{K}_{n-1}^l$  can be used to derive the mode shapes and the transformation matrix. This simplification slightly reduces the update rate of the experimental tangent stiffness matrix, which has been observed to be insignificant.

The experimental tangent stiffness matrix obtained using the above diagonalization approach can then be used in (6) to update the experimental restoring force used in numerical integration. Ideally, the results of this equation should match those found from the fitted polynomials used to estimate force increments. However, the calculation of forces through (6) ensures the use of the

most accurate data, and that the symmetry and positive-definiteness of the stiffness matrix is considered in calculation of the restoring forces, as described above.

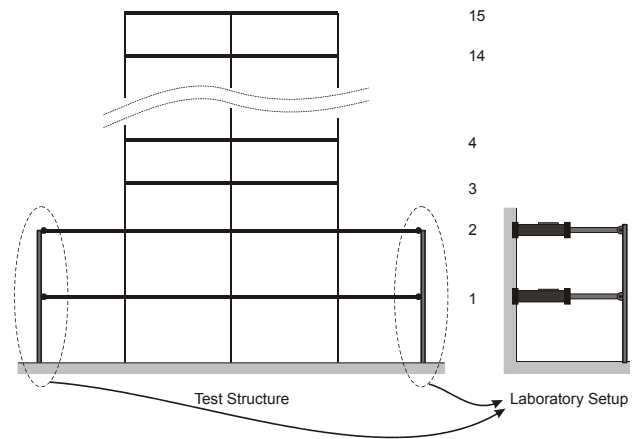
### Simulation Continuity

Convergence of iterations cannot be guaranteed in integration of nonlinear problems, particularly when they involve both experimental and numerical errors. Since the experimental substructures may have already been damaged, the simulation cannot be restarted due to convergence failure. Hence, the unconverged solutions need to be handled using an alternate approach to maintain the continuity of the simulation. In the integration algorithm presented in this paper, it is proposed to revert to the solution of operator-splitting method using numerical and experimental tangent stiffness matrices to complete integration steps where convergence fails. In this case, the states are updated only once, and instead of using fitted polynomials, actual increments of displacements and forces are used to update the experimental tangent stiffness matrix. It has been observed that the accuracy and stability of the simulation are not significantly affected when the integration steps with the above alternative solution are sparsely distributed through the simulation (Ahmadizadeh 2007).

## EXPERIMENTAL VERIFICATIONS

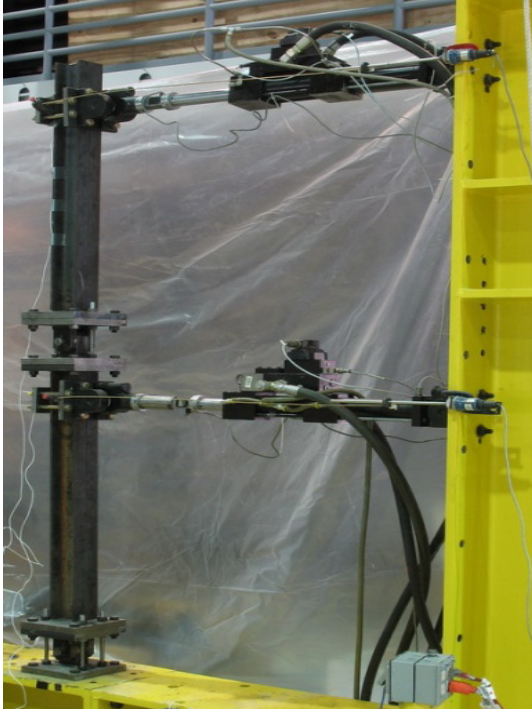
In this section, the effectiveness of the proposed integration algorithm is demonstrated through an actual hybrid numerical and experimental simulation. The hybrid simulation test system at Structural Engineering and Earthquake Simulation Laboratory (SEESL) at University at Buffalo consists of the test setup, actuator controllers, simulation host PC (running Simulink® (The MathWorks® Inc 1994-2007)), and xPC target computers (real-time environments). For fast online simulations, the simulation components communicate through Shared Common Random Access Memory Network (SCRAMNet). The actuator commands and measurements in this hybrid simulation system are updated at a rate of 1024 Hz.

The test structure, shown in Fig. 2, is a 15-story shear building with a setback at the third story level. The experimental substructure is selected as the two-story exterior columns of the lower levels. The connections of these columns to the rigid floors are assumed to be pinned. The remaining part of structure is modeled numerically. Bouc-Wen hysteretic models (Bouc 1967; Wen 1976) are selected to relate the story shear and story drifts of the numerical substructure. The structure is assigned a stiffness of 8.86 kN/mm with a yield displacement of 3.6 mm with the exception of the first story with a stiffness of 1.77 kN/mm. Note that the experimental substructures will provide additional stiffness to the first two stories. The floor weights are selected to achieve a fundamental period of 1.0 s. The structure is assumed to be viscously damped with a damping ratio of 2% of critical.



**Fig. 2. Fifteen story hybrid structural model used in experimental simulations with experimental substructures indicated**

The two-degree-of-freedom experimental substructure representing the two-story columns is shown in Fig. 3. The specimen consists of two columns and two clevises mounted on top of each other. The lateral resistance at the clevis is provided using replaceable coupons that can be inserted in the clevises. The columns are designed to remain elastic, thus limiting the yielding to the coupons for low-cost nonlinear simulations.



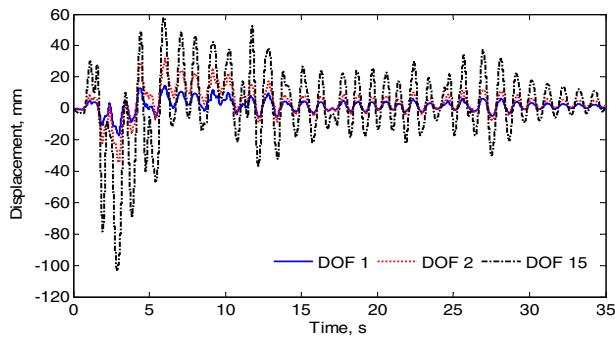
**Fig. 3. Two-degree of freedom experimental substructure consisting of cantilever with two horizontal actuators**

the coupons for low-cost nonlinear simulations.

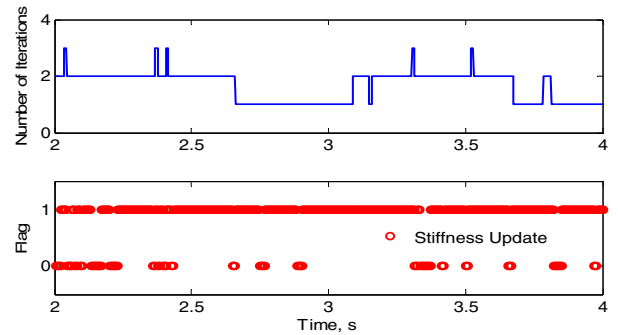
It should be noted that the experimental substructure behavior is governed by rotations at clevises or flexure of columns, rather than shear deformations. The delay was measured to be about 15 ms in both actuators, and was compensated using Equation (5) with an adjusted time step (Ahmadizadeh et al. 2007).

The response of the test structure is simulated for 1940 El Centro earthquake (peak ground acceleration = 0.32g) with an integration time step of 10/1024 s. The displacements at two lower stories along with the top floor displacements are shown in Fig. 5. This graph shows that the structure undergoes a small permanent deformation. In this simulation, all of the integration steps were completed with successful iterations (all iterations converged before the maximum number of iterations). However, the experimental tangent stiffness matrix was

updated in 69.5% of integration steps. In other integration steps, the displacement increments were small, or estimation of forces corresponding to iterative displacements failed within the predetermined time variation range. The number of iterations and steps with successful experimental stiffness matrix updates are shown in Fig. 4 during a short period of simulation. In this figure, the flag variable shown as small circles takes values of 0 or 1, showing the status of experimental stiffness matrix update. As illustrated, convergence is normally achieved after only



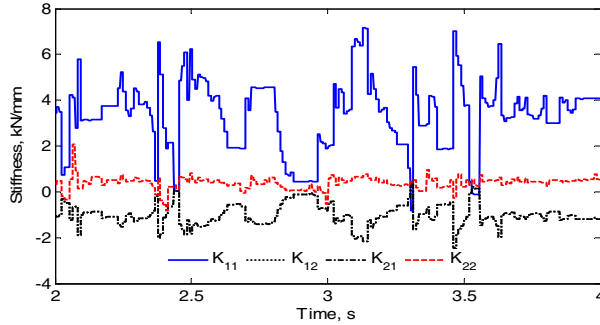
**Fig. 5. Displacement response of hybrid numerical and experimental model subjected to 1940 El Centro earthquake**



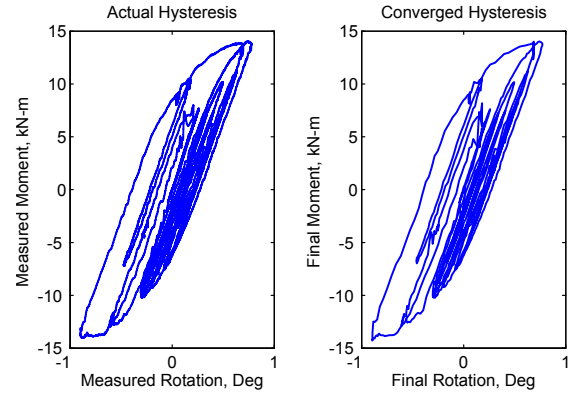
**Fig. 4. Number of iteration in each integration step and indication of steps providing updated experimental tangent stiffness**

two iterations, occasionally requiring three iterations. The number of iterations may be more in larger structures, or structures with more complex behavior.

The elements of the estimated experimental tangent stiffness matrix are shown in Fig. 6 during a short period of simulation. As illustrated, the estimates contain a fair amount of noise, which is far smaller than when displacement increment threshold is not taken into account. The noise may be further reduced by using better instrumentation, or using equivalent low-pass filters on force and displacement measurements, taking care not to alter their phase agreement. A detailed



**Fig. 6. Time variation of stiffness matrix elements during a short time window of the simulation**



**Fig. 7. Hysteretic behavior of lower plastic hinge and difference between actual measured behavior and behavior observed by numerical model based on final converged displacement.**

examination of Fig. 6 indicates that the stiffness reductions are gradual and coincide with large drifts. On the other hand, stiffness matrix elements are shown to sharply increase in displacement reversals, which correspond to elastic recovery.

As a measure of accuracy, the hysteretic behavior of the first story hinge is shown in Fig. 7 using two different sets of data. The actual hysteretic behavior of this hinge is plotted using measurements without any modifications, that may be performed on the measurements for error compensation or signal correction.

The converged hysteretic behavior of the first story hinge is also shown in Fig. 7 which is obtained using states at the end of the iterative corrections. Through these iterations, the estimated tangent stiffness of the experimental substructure is used to modify the experimental restoring force and update the states. Hence, the fact that the converged hysteretic behavior does not show any significant distortion from the actual behavior demonstrates the effectiveness of the proposed procedure for estimation of the experimental stiffness matrix and restoring forces.

## CONCLUSIONS

An iterative implicit integration procedure was proposed for hybrid simulation. Through polynomial interpolations and extrapolations, this integration method eliminates the need for physical application of the iterative displacements on the experimental substructure. In this integration method, the tangent stiffness matrix of the numerical substructure can be determined



using the conventional methods of numerical simulation. An efficient method was also proposed for online estimation of experimental tangent stiffness matrix. This method requires minimal information about the experimental substructure, and only uses the readily-available force and displacement measurements to update the stiffness matrix. Using this method, the estimation of experimental tangent stiffness matrix is carried out only in steps with significant displacement increments, to avoid the excessive contamination of results by experimental errors and measurement noise. An alternative solution procedure was also adopted to complete integration steps with failed iterations and maintain the continuity of the simulation. The proposed integration method together with the procedure introduced for estimation of experimental tangent stiffness was shown to have excellent performance in accurately capturing the behavior of the experimental substructure and maintaining the energy balance of the simulation for testing relatively large and nonlinear structures. Future studies will examine the implementation of this approach in finite element software, allowing for more complex modeling of the numerical model.

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